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Exact solutions of a modified fractional diffusion equation in the finite and semi-infinite domains

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HIGHLIGHTS

- We investigate the solutions for the modified fractional diffusion equation.
- Exact semi-infinite and finite solutions subject to absorbing boundaries are found.
- The crossover between more and less anomalous behaviour is demonstrated.

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ABSTRACT

We investigate the solutions of a modified fractional diffusion equation which has a secondary fractional time derivative acting on a diffusion operator. We obtain analytical solutions for the modified equation in the finite and semi-infinite domains subject to absorbing boundary conditions. Most of the results have been derived by using the Laplace transform, the Fourier Cosine transform, the Mellin transform and the properties of Fox *H* function. We show that the semi-infinite solution can be expressed using an infinite series of Fox *H* functions similar to the infinite case, while the finite solution requires double infinite series including both Fox *H* functions and trigonometric functions instead of one infinite series. The characteristic crossover between more and less anomalous behaviour as well as the effect of absorbing boundary conditions are clearly demonstrated according to the analytical solutions.

1. Introduction

Diffusion is one of the most important phenomena encountered in numerous physical, chemical and biological systems [1]. However, the picture that has emerged over the last few decades clearly reveals that an increasing number of natural phenomena do not fit into the relatively simple description of normal diffusion [2]. Anomalous diffusion turns out to be quite ubiquitous and it is characterized by a nonlinear behaviour for the mean square displacement in the course of time [3].

The actual reason or the very nature of anomalous diffusion may vary a lot and there are many approaches or frameworks such as fractional partial differential equations and continuous time random walk models that can be used to describe these anomalous diffusion processes [4]. Depending on the underlying mechanism of anomalous diffusion, various fractional

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model equations have been proposed to account for the characteristic behaviour of anomalous diffusion. For example, the fractional Fokker–Planck equation has been introduced for the description of anomalous transport in the presence of an external field [5]. In addition, more generalized fractional diffusion equations that usually contain a mix of nonlinear terms and space- or time-fractional derivatives have been extensively investigated due to their broadness of applications [6–13]. In all cases, it is always of interest to find exact analytical solutions subject to certain initial and boundary conditions if not impossible.

Recently, some modified fractional diffusion models have been proposed to describe processes that become less anomalous as time progresses by the inclusion of a secondary fractional time derivative acting on a diffusion operator [14–18],

$$\frac{\partial}{\partial t}P(x,t) = \left(K_\alpha \frac{\partial^{1-\alpha}}{\partial t^{1-\alpha}} + K_\beta \frac{\partial^{1-\beta}}{\partial t^{1-\beta}}\right)\frac{\partial^2}{\partial x^2}P(x,t), \tag{1}$$

where $0 < \alpha < \beta \le 1$ and K_α, K_β are positive constants. In fact, this modified fractional diffusion equation is a special case of the more generalized distributed or variable order fractional diffusion equations which are employed to precisely depict the decelerating or accelerating diffusion processes [19,20]. Particularly, we mention that for the multi-term time-fractional diffusion-wave equation with constant or variable coefficients, exact solutions subject to different initial and boundary conditions have been given in form of the Fourier series via the multivariate or multinomial Mittag-Leffler function [21–24]. Some numerical methods have also been proposed for such initial value problems in the general nonlinear case [25].

A possible application of the modified fractional diffusion equation with two time scales is in econophysics and particularly the crossover between more and less anomalous behaviour has been observed in the volatility of some share prices [17]. In general, the modified equation is advantageous when describing processes which get less subdiffusive in the course of time. Note that the Riemann–Liouville fractional derivative operator is defined by

$$\frac{\partial^{1-\alpha}}{\partial t^{1-\alpha}}P(x,t) = \frac{1}{\Gamma(\alpha)}\frac{\partial}{\partial t}\int_0^t (t-\tau)^{\alpha-1}P(x,\tau)\mathrm{d}\tau. \tag{2}$$

At this point we note that Henry and Wearne [18] find in their derivation of fractional reaction–diffusion equations an additional term

$$\mathscr{L}^{-1}\left[\left(K_\alpha \frac{\partial^{-\alpha}}{\partial t^{-\alpha}} + K_\beta \frac{\partial^{-\beta}}{\partial t^{-\beta}}\right)\frac{\partial^2 P}{\partial x^2}\right]_{t=0}, \tag{3}$$

on the right of Eq. (1) where $\mathscr{L}^{-1}$ is the Laplace inverse transform. The value of this term is unclear as it necessitates the behaviour of the term to be known near $t = 0$. Recall that the Laplace transform of the Riemann–Liouville fractional derivative is given by

$$\mathscr{L}\left[\frac{\partial^{1-\alpha}}{\partial t^{1-\alpha}}Y(t)\right](s) = s^{1-\alpha}\mathscr{L}[Y(t)](s) - \left[\frac{\partial^{-\alpha}}{\partial t^{-\alpha}}Y(t)\right]_{t=0}, \quad 0 < \alpha \le 1. \tag{4}$$

Taking the Laplace transform of Eq. (1), we obtain

$$sP(x,s) - P(x,0) = \left(s^{1-\alpha}K_\alpha + s^{1-\beta}K_\beta\right)\frac{\partial^2}{\partial x^2}P(x,s) - \left[\left(K_\alpha \frac{\partial^{-\alpha}}{\partial t^{-\alpha}} + K_\beta \frac{\partial^{-\beta}}{\partial t^{-\beta}}\right)\frac{\partial^2}{\partial x^2}P(x,t)\right]_{t=0}. \tag{5}$$

Note if we include the additional term then the last term in Eq. (5) cancels. However, if we do not include the additional term, it can be shown from the solution of Eq. (1) that these terms are zero and can be neglected [17]. In addition, this extra term can also be absorbed into the initial condition. Therefore, we will not consider the additional term during our derivation for the same reason.

For this modified fractional diffusion equation, Langlands et al. obtained the solution with the form of an infinite series of Fox *H* functions in an infinite domain [17]; Liu et al. discussed the numerical method and analytical technique of the modified anomalous sub-diffusion equation with a nonlinear source term [26,27]; Chen and A. Mohebbi et al. developed different numerical methods to solve the two-dimensional modified diffusion equation [28,29]. It is also worth to note that similar fractional cable model equations have been introduced for modelling electrodiffusion of ions in nerve cells when molecular diffusion is anomalous subdiffusion due to binding, crowding or trapping [30]. Exact analytic solutions of the fractional cable equations have been derived in the infinite, semi-infinite and finite domains for different boundary conditions using Fourier and Laplace transform methods [31,32]. For the purpose of this paper, we will only consider the solutions for the modified fractional diffusion equation in the finite and semi-infinite domains subject to absorbing boundary condition.

2. Finite solution

For the case of a finite domain, the problem to be solved can be recast as a boundary value problem with the following boundary and initial conditions

$$P(-\ell,t) = 0, \qquad P(\ell,t) = 0, \qquad P(x,0) = \delta(x), \quad \ell > 0, \tag{6}$$

where δ is the Dirac delta function and $x = 0$ is the starting point of diffusion containing the initial concentration of the distribution.