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Characterization results based on non-additive entropy of order statistics

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HIGHLIGHTS

- We propose the measure of Tsallis entropy of order statistics and prove that it characterizes the underlying distribution function uniquely.
- We study this measure for various distributions used in lifetime testing and survival analysis.
- Also we define the measure of residual Tsallis entropy of order statistics and study some characterization results for it.

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ABSTRACT

We consider a non-additive generalization of the Shannon entropy measure using order statistics and show that this entropy measure characterizes the distribution function uniquely. Further we propose a residual non-additive entropy measure for order statistics and prove a characterization result for that also.

1. Introduction

Information theory and Statistical mechanics are closely related. Jaynes [1] has shown that one can consider statistical mechanics as statistical inference rather than a physical theory and get some important results using maximum entropy principle. The central idea of information theory revolves around the concept of uncertainty introduced by Shannon in 1948. Shannon entropy [2] for a non-negative continuous random variable X is given by

$$H(f) = H(X) = -\int_0^\infty f(x)\log f(x)\mathrm{d}x, \tag{1}$$

where $f(x)$ is the pdf of a random variable X. The measure (1) is additive in nature in the sense that for two independent random variables X and Y

$$H(X * Y) = H(X) + H(Y),$$

where $X * Y$ denotes the joint random variable.

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A well known generalization of (1) is Tsallis entropy [3] which is given by

$$H_\beta(X) = \frac{1}{1-\beta}\left[\int_0^\infty f^\beta(x)\mathrm{d}x - 1\right], \quad \beta > 0,\ \beta \neq 1. \tag{2}$$

Clearly as $\beta \to 1$, (2) reduces to (1). This entropy is similar to Shannon entropy except for its non-additive nature, that is, in this case

$$H_\beta(X * Y) = H_\beta(X) + H_\beta(Y) + (1-\beta)H_\beta(X)H_\beta(Y).$$

In general, the non-additive measures of entropy find justifications in many physical, biological and chemical phenomena. Some properties and applications of (2) have been studied in Ref. [4]. Wilk and Woldarczyk [5] have demonstrated a situation where uncertainty about an object or a system is calculated by Tsallis entropy as Shannon entropy failed to provide uncertainty. The measure (2) has also been extensively used in image processing and signal processing, refer to Refs. [6,7]. Also one of the important applications of order statistics is to construct median filters for image and signal processing. Considering importance of Tsallis entropy and order statistics, we try to extend the concept of Tsallis entropy using order statistics which can be further used in image or signal processing.

Suppose $X_1, X_2, \ldots, X_n$ are n independent and identically distributed observations from a distribution F, where F is differentiable with a density f which is positive in an interval and zero elsewhere. The order statistics of a sample is arrangement of $X_1, X_2, \ldots, X_n$ from the smallest to the largest which is denoted by $X_{1:n}, X_{2:n}, \ldots, X_{n:n}$. Order statistics has found numerous applications in day-to-day life. It is used in detection of outliers, characterizations of probability distributions, testing strength of materials, etc. for details refer to Refs. [8,4]. In reliability theory, order statistics has been used for statistical modeling, as the ith order statistics in a sample of size n corresponds to life length of an $(n-i+1)$-out-of-n system.

Several authors have studied information theoretic properties of an ordered data. Wong and Chen [9] have shown that the difference between the entropy of the ith order statistics and the average entropy is a constant. Some recurrence relations for entropy of an ordered sequence have been proved by Park [10]. Various properties of Shannon entropy of the ith order statistics are discussed widely in Refs. [11,12]. We extend the work done so far by studying it for the non-additive entropy (2).

The paper is organized as follows. In Section 2, we propose a non-additive entropy for the ith order statistics and study a characterization result for this entropy along with a few specific examples. Section 3 considers the residual non-additive entropy for the ith order statistics and a characterization result for this. Section 4 contains conclusion and comments.

2. Non-additive entropy for the *i*th order statistics

Analogous to (2), non-additive entropy for the ith order statistics $X_{i:n}$, $1 \le i \le n$, is proposed as

$$H_\beta(X_{i:n}) = \frac{1}{1-\beta}\left[\int_0^\infty f_{i:n}^\beta(x)\mathrm{d}x - 1\right], \quad \beta > 0,\ \beta \neq 1, \tag{3}$$

where $f_{i:n}(x)$ is the pdf of the ith order statistics given by

$$f_{i:n}(x) = \frac{F(x)^{i-1}(1-F(x))^{n-i}f(x)}{B(i, n-i+1)}, \tag{4}$$

here $B(i, n-i+1)$ is the beta function with parameters i and $(n-i+1)$, for details see Ref. [13]. For $n = 1$, which implies $i = 1$ also, (3) reduces to (2), and when $\beta \to 1$, (3) reduces to the Shannon entropy of the ith order statistics studied in Ref. [12].

2.1. Characterization result

In this section we prove an important aspect that the proposed non-additive entropy for the ith order statistics characterizes the distribution function of the parent random variable uniquely. Before proving the main result, we state the following lemma which is a corollary to Stone–Weierstrass Theorem.

Lemma 2.1. *If η is a continuous function on $[0, 1]$ such that $\int_0^1 x^n\eta(x)\mathrm{d}x = 0$, for all $n \ge 0$, then $\eta(x) = 0$, for all $x\varepsilon[0, 1]$.*

Next, we prove the following characterization theorem.

Theorem 2.1. *Let X and Y be two non-negative random variables having the common support. Let $H_\beta(X_{i:n}) < \infty$ and $H_\beta(Y_{i:n}) < \infty$ be their non-additive entropies for the ith order statistics respectively. Then X and Y belong to same family of distributions if, and only if, $1 \le i \le n$,*

$$H_\beta(X_{i:n}) = H_\beta(Y_{i:n}), \quad \forall n \ge i,$$

here n the sample size is a positive integer.