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Theoretical studies of self-organized criticality

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Abstract

These notes are intended to provide a pedagogical introduction to the abelian sandpile model of self-organized criticality, and its related models. The abelian group, the algebra of particle addition operators, the burning test for recurrent states, equivalence to the spanning trees problem are described. The exact solution of the directed version of the model in any dimension is explained. The model's equivalence to Scheidegger's model of river basins, Takayasu's aggregation model and the voter model is discussed. For the undirected case, the solution for one-dimensional lattices and the Bethe lattice is briefly described. Known results about the two dimensional case are summarized. Generalization to the abelian distributed processors model is discussed. Time-dependent properties and the universality of critical behavior in sandpiles are briefly discussed. I conclude by listing some still-unsolved problems.

Keywords: Abelian; Sandpile; Avalanches; Self-organized criticality; Dissipation; River networks

1. Introduction

These notes started as a written version of the lectures given at the Ecole Polytechnique Federale, Lausanne in 1998 [1]. I have updated and reorganized the material somewhat, and added a discussion of some more recent developments once before. The aim is to provide a pedagogical introduction to the abelian sandpile model and other related models of self-organized criticality. In the last nearly two decades, there has been a good deal of work in this area, and some selection of topics, and choice of level of detail has to be made to keep the size of notes manageable. I shall try to keep the discussion self-contained, but algebraic details will often be omitted in favor of citation to original papers. It is hoped that these notes will be useful to students wanting to learn about the subject in detail, and also to others only seeking an overview of the subject.

In these notes, the main concern is the study of the so-called abelian sandpile model (ASM), and its related models: the loop-erased random walks, the $q \to 0$ limit of the q-state Potts model, Scheidegger's model of river networks, the Eulerian walkers model, the abelian distributed processors model, etc. The main appeal of these models is that they are analytically tractable. One can explicitly calculate many quantities of interest, such as properties of the steady state, and some critical exponents, without too much effort. As such, they are very useful for developing our understanding of the basic principles and mechanisms underlying the general theory. A student could think of the ASM as a 'base camp' for the explorations into the uncharted areas of

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non-equilibrium statistical mechanics. The exact results in this case can also serve as proving grounds for developing approximate treatments for more realistic problems.

While these models are rather simple to define, and not too complicated to solve (at least partly), they are non-trivial, and cannot be said to be well understood yet. For example, it has not been possible so far to determine the critical exponents for avalanche distributions for the oldest and best known member of this class: the undirected sandpile model in two dimensions. There are many things we do not understand. Some will be discussed later in the lectures.

Our focus here will be on the mathematical development of these models. However, it is useful to start with a brief discussion of their origin as simplified models of physical phenomena.

2. Self-organized criticality in nature

It was the great insight of Mandelbrot that fractals are not mathematical curiosities, but that many naturally occurring objects are best described as fractals. Examples include mountain ranges, river networks, coastlines, etc. His book [2] remains the best introduction to fractals, for its clear exposition, evocative pictures, and literary style. The word 'fractal structure' here means that some correlation functions show non-trivial power law behavior. For example, in the case of mountain ranges, the irregular and rough height profile can be characterized by how $\Delta h(R)$, the difference of height between two points separated by a distance R, varies with R. It is found that

$$\langle [\Delta h(R)]^2 \rangle \sim R^x, \tag{1}$$

where $\langle\rangle$ denotes averaging over different spatial points at the fixed horizontal separation R, and x is some non-trivial power. When measured, the exponent x seems to not vary much between one mountain range and another.

In the case of river networks, the fractal structure can be characterized in terms of the empirical Hack's law [3]. This law describes how the catchment area of any particular stream in a river basin grows as we go down-stream along the river. If A is the catchment area, and ℓ is the length of the principal stream up to this point, Hack's law states that on the average A grows as ℓ^y, where y is some exponent, $y \simeq 1.6$.

In other examples the fractal behavior is found not in the spatial structure itself, but is manifest the power-law dependence between some physical observables. For example, in the case of earthquakes, the frequency of earthquake of total energy E is found to vary as E^{-z}, where z is a number close to 2, for many decades of the energy range. (This is the well-known Gutenberg–Richter law [4]). Recently, it was shown that the frequency of rain of a given intensity (amount of rain per unit area in a short time interval at the observation site) also follows a Gutenberg–Richter like power law for about five orders of magnitude of intensity [5].

An example in a similar spirit is that of fluid turbulence where an example of the fractal behavior is the way mean squared velocity difference scales with distance, or in the spatial fractal structure of regions of high dissipation [6].

A good understanding of these fractal structures should allow us to calculate the values of the critical exponents from physical principles, and not just tabulate them from experimental data. As the systems involved have many interacting degrees of freedom, techniques of statistical physics are expected to be useful in this.

Systems exhibiting long-ranged correlations with power law decay over a wide range of length scales are said to have critical correlations. This is because correlations much larger than the length-scale of interactions were first studied in equilibrium statistical mechanics in the neighborhood of a critical phase transition. In order to observe such critical phenomena in equilibrium systems, one needs to fine-tune some physical parameters (such as the temperature and pressure) to specific critical values, something rather unlikely to occur in a naturally occurring process such as the formation of mountains, where the actual shape is a result of competing processes of plate tectonics and erosion, and the external variables like the temperature have undergone big changes in time. The systems we want to study may be said to be in a steady state as while there is variation in time, overall properties are roughly unchanged over the time scale of observation. However, these systems are not in equilibrium: they are open and dissipative systems which require input of energy from outside at a constant rate to offset the dissipation. We define such states to be *non-equilibrium steady states*.