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Hurst exponents for interacting random walkers obeying nonlinear Fokker–Planck equations

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ABSTRACT

Anomalous diffusion of random walks has been extensively studied for the case of non-interacting particles. Here we study the evolution of nonlinear partial differential equations by interpreting them as Fokker–Planck equations arising from interactions among random walkers. We extend the formalism of generalized Hurst exponents to the study of nonlinear evolution equations and apply it to several illustrative examples. They include an analytically solvable case of a nonlinear diffusion constant and three nonlinear equations which are not analytically solvable: the usual Fisher equation which contains a quadratic nonlinearity, a generalization of the Fisher equation with density-dependent diffusion constant, and the Nagumo equation which incorporates a cubic rather than a quadratic nonlinearity. We estimate the generalized Hurst exponents.

1. Introduction

Random walks and anomalous diffusion [1–5] have traditionally been approached from the perspective of whether or not they include memory. Markoffian processes describe random walks with short (e.g., exponentially decaying) memory, whereas non-Markoffian walks describe the general case of random walks with memory of the (possibly complete) history [6]. Less work has gone towards the investigation of how interactions among random walkers can change the global behavior. The presence of even local interactions can dramatically alter global behavior. For example, when the diffusion equation for Brownian motion governed by a Wiener process is augmented with a convective term whose strength is linear in the local density of the walkers, the resulting Burgers equation leads to qualitatively different behavior: the principle of superposition breaks down and Gaussian solutions become unstable. A recent study [7] has generalized the formalism of Hurst exponents to address the problem in the context of Burgers equation arising from hydrodynamic models of vehicular traffic flow. Specifically, it was shown that initial conditions become important due to the breakdown of the principle of linear superposition. In the present paper we extend that study to several further cases of interacting random walkers.

The Hurst exponent quantifies how quickly particles diffuse. For the case of zero drift velocity, the Hurst exponent H describes how the mean squared displacement of a random walker $\langle x^2 \rangle \sim t^{2H}$ scales with time t. Normal diffusion gives $H = 1/2$ due to the central limit theorem, which guarantees convergence of the probability density function of the walkers's position to a Gaussian. For non-interacting particles, the Fokker–Planck equation for the probability density $P(x, t)$ of a particle is linear in P. Hence, the propagator of the Fokker–Planck equation contains all the relevant information concerning H.

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However, when particles interact, the resulting evolution equation might be nonlinear; hence Green's functions and propagators will obviously not exist.

In Section 2 we recall and discuss the findings of Ref. [7] that the usual method of estimating Hurst exponents can lead to spurious predictions as a result of non-negligible effects of initial conditions. In Sections 3–6 we apply the method to study an exactly solvable system as well as a number of nonlinear diffusion equations, including the Fisher and Nagumo equations.

2. Hurst exponents for interacting random walkers

2.1. Generalized Hurst exponents

A number of methods can be applied to quantify different aspects of anomalous diffusion. Anomalous diffusion has been studied using a number of formalisms and approaches. Continuous time random walks [2,8] and generalized master equations [9] (GMEs) are formally equivalent [10]. Fractional partial differential equations [11,12] are equivalent to GMEs. Here we use the formalism of Hurst exponents, which can be related to Hölder exponents [13–16], describing the degree of differentiability along the trajectories. This formalism has been used to study the anomalous dynamics of different systems. Recently, this has been applied in the field of finance as well [17].

One can define the Hurst exponent $H(q)$ for a stationary stochastic process [14,16] in terms of the scaling of the absolute moments of the density:

$$\bar{x} \equiv \langle x \rangle \tag{1}$$

$$M_q(t) \equiv \langle |x - \bar{x}|^q \rangle \sim t^{qH(q)} \tag{2}$$

where the averages are taken over the propagator. Brownian motion and normal diffusion correspond to $H(q) = H = 1/2$, whereas anomalous diffusion corresponds to all other cases. As discussed in Ref. [7], one can generalize the concept to allow a scale dependence [18–20], such that $H = H(q, t)$:

$$M_q(t) \sim t^{qH(q,t)}. \tag{3}$$

For instance, the telegrapher's equation [21] has a mean squared displacement that grows quadratically for small times but linearly for larger times. The behavior is ballistic at small times, $(H(q, t) \approx 1)$ but diffusive at large times $(H(q, t) \to 1/2)$. This behavior can also be written in terms of the asymptotically defined Hurst exponent $H(q)$ and a scaling function f, such that $M_q(t) \sim t^{qH(q)} f(t/t^*)$, t^* being the typical crossover time and $H(q) = 1/2$ with $f \sim t$ for $t \ll t^*$ and $f \sim$ constant for $t \gg t^*$. However, we don't know, a priori, that such crossovers are generic, which limits the applicability of this scaling description. Also, for systems for which the Hurst exponent changes continuously in time, it is difficult to write proper scaling function and thus Eq. (3) gives a natural and more generic way to approach a problem.

2.2. Nonlinear Fokker–Planck equations

Standard methods of deriving Fokker–Planck equations from Langevin equations lead always to linear equations [7]. Consider, for example, the Boltzmann equation for gases, obeyed by the one-molecule distribution function. It is nonlinear, in contrast to the underlying linear Liouville equation for the N-molecule Liouville density. In the absence of inter-particle interactions, the Boltzmann equation would be linear. Similarly, if the gas molecules interact with a fixed system of random scatterers, the Boltzmann equation would be nontrivial but still linear. The standard manner of applying Hurst and Hölder exponents would work here. The case of interacting random walkers is different. If intermolecular interactions are turned on, nonlinearity enters the picture and immediately makes unavailable the superposition principle and propagator analysis. We recall the standard manner in which the Hurst exponent $H = H(2)$ is usually obtained from the behavior of the mean squared displacement for linear equations. In terms of the propagator or the Green function $\psi(x, x_0, t)$ and the initial distribution $P(x_0)$, the Hurst exponent is given by the scaling behavior of

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathrm{d}x\mathrm{d}x_0 \; (x - x_0)^2 \psi(x, x_0, t) P(x_0).$$

For a translationally invariant (homogeneous) system such as the one under consideration in this paper, this equation reduces to

$$\langle x^2 \rangle_\delta = \int \mathrm{d}x \, x^2 \psi(x, t) \tag{4}$$

because the propagator is a function of the difference $x - x_0$. We use the suffix δ in the left hand side of Eq. (4) to emphasize that the $\langle x^2 \rangle$ used here can be considered to be the one calculated for an initially localized initial condition $P(x_0) = \delta(x_0)$. This can also be used for a collection of many random walkers provided they are non-interacting among themselves: the initial distribution P_0 is irrelevant in a linear system of non-interacting particles.