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Phase-space Lagrangian dynamics of incompressible thermofluids

Marco Tessarotto [a,b,c], Claudio Cremaschini [d,e], Massimo Tessarotto [c,f,*]

[a] Department of Electronics, Electrotechnics and Informatics, University of Trieste, Italy
[b] Civil Protection Agency, Regione Friuli Venezia-Giulia, Palmanova (Udine), Italy
[c] Department of Mathematics and Informatics, University of Trieste, Trieste, Italy
[d] International School for Advanced Studies (SISSA), Trieste, Italy
[e] National Institute of Nuclear Physics (INFN), Trieste, Italy
[f] Consortium for Magnetofluid Dynamics, Trieste, Italy

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ABSTRACT

Phase-space Lagrangian dynamics in ideal fluids (i.e., continua) is usually related to the so-called *ideal tracer particles*. The latter, which can in principle be permitted to have arbitrary initial velocities, are understood as particles of infinitesimal size which do not produce significant perturbations of the fluid and do not interact among themselves. An unsolved theoretical problem is the correct definition of their dynamics in ideal fluids. The issue is relevant in order to exhibit the connection between fluid dynamics and the classical dynamical system, underlying a prescribed fluid system, which uniquely generates its time-evolution.

The goal of this paper is to show that the tracer-particle dynamics can be *exactly* established for an arbitrary incompressible fluid uniquely based on the construction of an inverse kinetic theory (IKT) [M. Tessarotto, M. Ellero, Bull. Am. Phys. Soc. 45 (9) (2000) 40; M. Tessarotto, M. Ellero, AIP Conf. Proc. 762 (2005) 108. RGD24, Italy, July 10–16, 2004; M. Ellero, M. Tessarotto, Physica A 355 (2005) 233; M. Tessarotto, M. Ellero, Physica A 373 (2007) 142, arXiv: physics/0602140; M. Tessarotto, M. Ellero, in: M.S. Ivanov, A.K. Rebrov (Eds.), Proc. 25th RGD, International Symposium on Rarefied gas Dynamics, St. Petersburg, Russia, July 21–28, 2006, Novosibirsk Publ. House of the Siberian Branch of the Russian Academy of Sciences, 2007, p. 1001, arXiv:physics/0611113; M. Tessarotto, C. Cremaschini, Strong solutions of the incompressible Navier–Stokes equations in external domains: Local existence and uniqueness, arXiv:0809.5164v1 [math-ph], 2008]. As an example, the case of an incompressible Newtonian thermofluid is considered here.

1. Introduction

The basic aspect of fluid dynamics is related to the definition of the Lagrangian dynamics which characterizes both compressible and incompressible fluids. The customary approach to the Lagrangian formulation is based typically on a configuration-space description, i.e., on the introduction of the (configuration-space) *Lagrangian path*, $\mathbf{r}(t)$, spanning the configuration-space (fluid domain) Ω. Here $\mathbf{r}(t)$ denotes the solution of the initial-value problem $\frac{D\mathbf{r}}{Dt} = \mathbf{V}(\mathbf{r}, t)$, with $\mathbf{r}(t_o) = \mathbf{r}_o$. Here $\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \mathbf{V} \cdot \nabla$ denotes the so-called "fluid" convective derivative, $\mathbf{r}_o$ an arbitrary vector belonging to $\overline{\Omega}$ and $\mathbf{V}(\mathbf{r}, t)$ the velocity fluid field, to be assumed continuous in $\overline{\Omega}$ (closure of Ω) and suitably smooth in Ω. However,

* Corresponding author at: Department of Mathematics and Informatics, University of Trieste, Trieste, Italy. Tel.: +39 040 558 2643; fax: +39 040 558 2666.

E-mail address: massimo.tessarotto@cmfd.univ.trieste.it (M. Tessarotto).

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in turbulence theory the statistical formulation for the associated joint probability density for velocity increments requires the introduction of a phase-space representation of suitable type [1], in which usually the phase-space is identified with $\Gamma = \Omega \times \mathbb{R}^3$ (with closure $\overline{\Gamma} = \overline{\Omega} \times \mathbb{R}^3$), here denoted as *restricted phase-space*. Therefore it is natural to seek possible phase-space representations of this type for fluid systems. The goal of this investigation is concerned with the formulation of *restricted phase-space Lagrangian dynamics* in such a way that the phase-space Γ coincides with the direct product space $\Gamma = \Omega \times V$, Ω being the fluid domain and V (velocity space) the set $\mathbb{R}^3$. In particular in this paper, extending the formulation previously developed (Cremaschini et al. [2] and Tessarotto et al. [3,4]), we shall adopt for this purpose a so-called *phase-space inverse kinetic theory (IKT)*(see also Tessarotto et al. [5–10]). Basic feature (of such an approach) it that is relies on first principles, i.e., classical statistical mechanics and a prescribed complete set of fluid equation. This permits us to advance in time the relevant fluid fields by means of *phase-space Lagrangian equations* defined by the vector field $\mathbf{X}(\mathbf{x}, t)$, namely

$$\begin{cases} \dfrac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{X}(\mathbf{x}, t), \\ \mathbf{x}(t_o) = \mathbf{x}_o, \end{cases} \tag{1}$$

where $\mathbf{x}_o$ is an arbitrary initial state of $\overline{\Gamma}$ (closure of the phase-space Γ). The result appears relevant in particular for the following reasons: (1) the Lagrangian dynamics here determined permits to advance in time self-consistently the fluid fields, i.e., in such a way that they satisfy identically the required set of fluid equations. For isothermal fluids, this conclusion is consistent with the results indicated previously [7]; (2) the Lagrangian dynamics takes into account the specific form of the phase-space distribution function which advances in time the fluid fields; (3) the theory permits an exact description of the motion of those particles immersed in the fluid which follow the Lagrangian dynamics (classical molecules).

Phase-space Lagrangian dynamics and particle dynamics in ideal fluids are closely related issues. In fact both must be uniquely described via a suitable complete set of fluid fields $\{Z(\mathbf{r}, t)\}$ which define the fluid state. This refers, in particular, to the so-called *ideal tracer particles*, for which both self-interaction produced by the perturbations of the fluid fields generated by the same particles and binary collisions among them are negligible (in this sense they can therefore be intended also as "collisionless"). It is well known, however, that in customary approaches (see for example Maxey and Riley [11]) the equations of motion for ideal tracer particle are only known in some approximate sense and therefore do not reproduce exactly the correct fluid dynamics.

The purpose of this paper is to show that *an exact solution* can be reached for phase-space Lagrangian dynamics, and in particular for the conventional tracer dynamics, based on the formulation of a suitable IKT. By definition, an IKT must provide the complete set of fluid equations describing the fluid, by means of velocity moments of an appropriate phase-space probability density function (pdf). We intend to show that such a theory can be uniquely determined in the framework of classical statistical mechanics by invoking suitable statistical assumptions on the IKT. In particular, we present here a theory which applies to incompressible Newtonian fluids, including both isothermal and non-isothermal fluids [2–4]. In the following we intend to show that customary tracer-dynamics equations due to several authors – including Tchen [12], Corrsin and Lumley [13], Buevich [14] and Riley [15] and Maxey and Riley [11] – are incompatible with the exact phase-space Lagrangian formulation here obtained.

In detail the plan of the paper is as follows. First, in Section 2 previous approaches to ideal tracer-particle dynamics are summarized. Second in Section 3 a comparison between the Eulerian and the Lagrangian phase-space approaches is provided. Furthermore, in Section 4 the IKT for incompressible thermofluids is presented. This permits us to determine the appropriate form of the vector field $\mathbf{X}(\mathbf{x}, t)$. Next, in Section 5 the Lagrangian formulation of IKT is discussed in detail. The new set of phase-space Lagrangian equations are shown to advance uniquely in time the relevant fluid fields of an incompressible thermofluid. As a basic consequence, in Section 6 we will derive the exact dynamics of ideal tracer particles (see below for definition), comparing it with previous results.

2. Previous approaches to tracer-particle dynamics

The motion of small particles (such as solid particles or droplets, commonly found in natural phenomena and industrial applications) which can be injected in a fluid with arbitrary initial velocity, in practice, may be very different from that of the fluid. The accurate description of particle dynamics, as they are pushed along erratic trajectories by binary collisions (in real fluids) and by fluctuations of the fluid fields (in ideal fluids), is fundamental to transport and mixing in turbulence [1]. It is essential, for example, in combustion processes [16], in the industrial production of nanoparticles [17] as well as in atmospheric transport, cloud formation and air-quality monitoring of the atmosphere [18,19]. The Lagrangian approach – denoted as the Lagrangian turbulence (LT) – has been fruitful in advancing the understanding of the anomalous statistical properties of turbulent flows [20]. In particular, the dynamics of particle trajectories has been used successfully to describe mixing and transport in turbulence [16,21]. Nevertheless, issues of fundamental importance remain unresolved (see for example Refs. [22,23] for recent results regarding the Lagrangian view of passive scalar turbulence). In the past, the treatment of the Lagrangian dynamics in turbulence was based on stochastic models of various nature, pioneered by the meteorologist Richardson [24] (see also Refs. [22,23]). These models, which are based on tools borrowed from the study of random dynamical systems, typically rely – however – on experimental verification rather than on first principles. However, in most cases there remains a lack of experimental data to verify the reliability of such models [23]. Verification can be based,