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General relativistic Boltzmann equation, I: Covariant treatment

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ABSTRACT

This series of two articles aims at dissipating the rather dense haze existing in the present literature around the General Relativistic Boltzmann equation. In this first article, the general relativistic one-particle distribution function in phase space is defined as an average of delta functions. Thereupon, the general relativistic Boltzmann equation, to be obeyed by this function, is derived. The use of either contravariant or covariant momenta leads to different, but equivalent, forms of the equation.

The results of the present article are covariant, but not manifestly covariant. The transition to a manifestly covariant treatment, on the basis of off-shell momenta, is given in the second article.

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1. Introduction

Non-equilibrium systems are systems in which some kind of transport takes place, that is, transfer of energy, mass, or any other particle property, from one place in a system to another.

Central in the theoretical description of dilute non-equilibrium systems stands the so-called *one-particle distribution function*, usually denoted by the symbol f. All macroscopic properties of a dilute non-equilibrium system, such as its pressure, its energy density, its temperature, its electrical conductivity, its viscosity, etc., can be expressed in terms of this function f. The function f, in turn, can be determined by solving an equation — often called a *transport equation*— that describes the behavior of the non-equilibrium system.

The most famous of these equations is the non-relativistic equation derived by Ludwig Boltzmann at the end of the 19th century. In the forties, Lichnerowich and Marrot generalized this equation to the realm of *special* relativity [1]. Israel [2] was one of the first to calculate relativistic transport coefficients on the basis this equation. He also was one of the first authors that wrote down a *general* relativistic version of the Boltzmann-equation [3]. It is the aim of this article to transfer the special relativistic Boltzmann theory to the realm of general relativity. To that end, both the *definition of* f and the *equation for* f have to be generalized.

The existing literature on the Boltzmann-equations usually starts directly from manifestly covariant equations [3–5]. This makes it difficult to see what is actually happening: the steps of the derivation are obscured by the manifestly covariant formalism. Therefore, we opted for an approach which is covariant but not *manifestly* covariant. We will perform the transition to manifestly covariant equations in a the sequel to the present article [6].

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The material is organized as follows:

Section 2 reviews the non-relativistic and special relativistic distribution functions f_{nr} and f_{sr}. In general relativity, it is not clear, a priori, whether one should use the contravariant momentum p^i or the contravariant momentum p_i as a variable in the distribution function. We therefore define two distribution functions, one in the seven dimensional (t, x^i, p^i)-space, and one in the seven dimensional (t, x^i, p_i)-space. We call them f and f_*, respectively. Both f, given by Eq. (27), and f_*, given by Eq. (4), are proven to be general relativistic scalars. Although the choices for f or for f_* are physically totally equivalent, the two sets of equations they lead to are sufficiently different to make mandatory a separate and full treatment for each choice. It turns out, however, that a treatment around f_* is substantially simpler, and, therefore, highly preferable.

In Section 3, the general relativistic counterparts of the Boltzmann equation, to be obeyed by the two afore-defined distribution functions f and f_*, are derived. For simplicity's sake, we only deal with non-quantal particles, and do not discuss Bose–Einstein or Fermi–Dirac statistics. The final equations for f and f_* are (55) and (80), respectively.

The present article is concerned with equations which are covariant, but not always manifestly covariant. In the next article, we will reformulate our results in a *manifestly* covariant way.

In the main text, one needs two-, three- and four-dimensional surface and volume elements in spacetime and momentum space. These volume elements are defined and related to one another in Appendix A. Appendix B recalls Stokes's theorem, while Appendix C is devoted to induced metrics. The Appendices A–C have a pedagogical character.

An explicit proof of the scalar character of $f_*(t, x^i, p_i)$ has been given in Section 2.3. The corresponding explicit proof of the scalar character of $f(t, x^i, p^i)$ is given in Appendix D.

We use a metric with signature -2.

2. One-particle distribution function

In this section we will be concerned with a general relativistic generalization of the special relativistic one-particle position–momentum distribution function, $f_{sr}(t, x^i, p^i)$, $(i = 1, 2, 3)$. It will turn out that in general relativity there are *two* possible generalizations, which we will denote $f(t, x^i, p^i)$ and $f_*(t, x^i, p_i)$, both of which are general relativistic scalars.

Here and elsewhere, a lower asterisk at some quantity indicates that covariant momentum components (like p_i or q_i) are used as variables rather than contravariant ones (like p^i or q^i).

At the one hand, the distribution function $f(t, x^i, p^i)$ depends on variables that are directly measurable, just as in special relativity, namely the time–position (t, x^i) and the contravariant momentum p^i. In general relativity, however, this choice of variables leads to a quite intricate, and, hence, undesirable definition of f, namely the definition (27). It contains, next to the usual Dirac delta-functions in position and three-momentum space, factors containing the energy and the determinant of the metric.

The distribution function $f_*(t, x^i, p_i)$, on the other hand, contains covariant momentum variables, variables that depend on the metric via $p_i = g_{i\mu}(t, x^i)p^\mu$ $(\mu = 0, 1, 2, 3)$, which, therefore, are not directly measurable. However, the definition of the distribution function $f_*(t, x^i, p_i)$ in terms of Dirac delta-functions, given in Eq. (4), is simpler than the corresponding definition (27) for $f(t, x^i, p^i)$. Now, there are no extra factors like the energy or the determinant of the metric. As a consequence, the Boltzmann equation (80) for f_* takes a simpler form than the Boltzmann equation (55) for f.

All this entails the question: which one to choose? Since both seem have their intrinsic advantages, we treat both f and f_* and derive the Boltzmann-equations they should obey. As a running start, we take the non-relativistic and the special relativistic distribution functions, f_{nr} and f_{sr}, respectively.

2.1. Non-relativistic distribution function

Let us start by recalling the classical, non-relativistic definition of the one-particle position–momentum distribution function $f_{nr}(t, x^i, p^i)$. It is a function of the time t, a position vector $x^i \equiv (x^1, x^2, x^3)$ and a momentum vector $p^i \equiv (p^1, p^2, p^3)$. By definition, the combination $f_{nr}(t, x^i, p^i)\ \Delta^3x\Delta^3p$ yields the *average* number of particles which, in a non-equilibrium fluid characterized by a certain set of macroscopic variables, will be found in a small but finite volume element Δ^3x around the point x^i with momenta in a small but finite volume element Δ^3p around p^i. The spatial volume elements Δ^3x are supposed to be large enough to contain many particles, but small enough in order to make it possible to treat the distribution function as a constant all over these volume-elements.

Consider a large, macroscopic system of identical particles numbered $r = 1, 2, 3, \ldots$. Let $x_r^i(t)$ describe the trajectory of the r-th particle of the system. Let $p_r^i(t)$ be the momentum of the r-the particle at time t. Now, let us consider the expression

$$\sum_r \delta^{(3)}(x^i - x_r^i(t))\delta^{(3)}(p^i - p_r^i(t)) \tag{1}$$

where the symbol $\delta^{(3)}$ denotes a three-dimensional Dirac distribution, i.e., a product of three Dirac delta-functions, each of which contains a component of the vector x^i or p^i $(i = 1, 2, 3)$. Integration of this expression with respect to the space and momentum volumes Δ^3x and Δ^3p yields the number of particles which, at time t, are found within the volume element Δ^3x around the point x^i with momenta within the volume element Δ^3p around the point p^i. Hence, the expression (1) gives the particle density in (x^i, p^i)-space.

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