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On the irreconcilability of Pareto and Gibrat laws

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ABSTRACT

If business firms face a multiplicative growth process in which their growth rates are Laplace distributed and independent from their sizes, the size cannot be distributed according to a stationary Pareto distribution. Recent contributions, using formal arguments, seem to contrast with these statements. We prove that the proposed formal results are wrong.

1. Introduction

Among the most popular, tested and contended hypotheses about the growth process of business firms, one can, beyond any doubt, include the so called Gibrat's Law of Proportionate Effect, which postulates that firm growth rate is independent from firm size, and the Law of Pareto, which assumes a stationary power-like behavior of the upper tail of the size distribution of firms.

Recent contributions propose to extend the number of these "Laws". The hypothesis that the (log) growth rates are distributed according to a Laplace, originally proposed in Ref. [1], has received quite some attention in the related literature, and its validity has been extensively tested in different countries and for different data (see the empirical review in Ref. [2] and the theoretical investigation in Ref. [3]).

At the same time, in Ref. [4] it has been claimed that the joint distribution of firm sizes in two subsequent time steps is characterized by a "time symmetry" which makes the growth process reversible (a similar argument was previously put forward for the income distribution in Ref. [5]). Always in Ref. [4], on the basis of the observed time symmetry, the authors maintain that the Gibrat's Law of Proportionate Effect is consistent with the Pareto Law. Based on this result, in Ref. [6] it is suggested that not only can these two Laws be fulfilled at the same time, but that the Pareto behavior of the upper tail of the size distribution is in fact responsible for the observed Laplace shape of the growth rates distribution.

In the present paper it is shown, through formal arguments, similar to those introduced in Refs. [5,6], that the Laplace distribution of growth rates does not guarantee the contemporaneous fulfillment of both the Gibrat's and the Pareto's Laws, and also that none could. Indeed, the Law of Pareto and the Law of Gibrat cannot, in any respect, be reconciled: ignoring entry and exit dynamics, if firms face a multiplicative growth process in which their growth rates are independent from their sizes, then these sizes cannot be distributed according to a stationary Pareto distribution. Moreover, the time-symmetry of the bivariate probability distribution of firms size claimed in Ref. [4] is, by itself, incompatible with the Gibrat's Law.

In Section 2 we introduce a formal framework, largely borrowed from Ref. [4], that allows one to express the previous statements in their whole generality. Then, in Section 3, we will present the main points of this paper, that is we will discuss

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the relationship between Gibrat's Law and time symmetry of the size distribution, and the relationship between Laplace growth rates distribution and Pareto's Law. The obtained results are finally summarized in Section 4.

2. Formal definition of properties

Abstracting from precise economic definitions, let $\mathbf{S_1}$ and $\mathbf{S_2}$ be two random variables describing the size of the firm at two consecutive times, 1 and 2, respectively. Let $p_{1,2}(S_1, S_2)$ stand for the joint probability density that a firm has size S_1 at the first time and S_2 at the second. In what follows, we will avoid a complete specification of the firm growth process. Indeed our results will be based exclusively on the properties of the density function $p_{1,2}$ and its marginals. As such, these results apply both to discrete time models, more common in the literature, and to continuous time models [7].

Consider the size return over the two periods $\mathbf{R} = \mathbf{S_2}/\mathbf{S_1}$. By the simple rule of change of variables, the joint probability density $p_{1,R}$ of the initial size $\mathbf{S_1}$ and the return $\mathbf{R}$ reads

$$p_{1,R}(S_1, R) = S_1\, p_{1,2}(S_1, S_1\, R). \tag{1}$$

If $p_1(S_1)$ is the marginal distribution of the initial size, by Bayes rule, one has

$$p_{1,R}(S_1, R) = Q(R|S_1)\, p_1(S_1), \tag{2}$$

where Q stands for the conditional probability of a firm to grow at a rate equal to R when its initial size is S_1. Analogously, one can write

$$p_{1,R}(S_1, R) = P(S_1|R)\, p_R(R), \tag{3}$$

where p_R is the marginal distribution of return and P is the distribution of initial size conditional on the value of the return R. Using these definitions, it is possible to express, in rather general terms, the properties mentioned above.

The Pareto Law states that the upper tail of the unconditional firms size distribution (or the distribution of a number of other economic variables, see Ref. [8] for a partial account) follows a power-like behavior. In terms of probability density, one has the following

Property 1 (*Pareto Law*). *If sufficiently large firms are considered, the probability density of their size decreases with some inverse power of the size itself*

$$p_1(x) = Ax^{-\mu-1} \quad x > S_{\min}, \tag{4}$$

where $\mu > 0$ is known as the Pareto coefficient and $A > 0$ is a normalization constant.

The second property, which is known as Gibrat's Law or "Law of proportionate effect" [9], states the independence of growth rate from initial size. It reads

Property 2 (*Gibrat Law*). *The firm's growth rate is independent on this size. That is*

$$Q(R|S_1) = Q(R). \tag{5}$$

The third property concerns the Laplace shape of the growth rates distribution. It states that the probability density of log-return $r = \log(R)$ behaves like a symmetric exponential. In terms of the return R one has

Property 3 (*Laplace Distribution*). *The growth rate unconditional density $p_R(R)$ reads*

$$p_R(R) = \begin{cases} 2a\, R^{-a-1} & \text{if } R > 1 \\ 2a\, R^{a-1} & \text{if } R \leq 1, \end{cases} \tag{6}$$

where $a > 0$ is a scale parameter.

Finally, we consider a fourth property which imposes a symmetric structure on the joint size distribution. Formally one has

Property 4 (*Symmetry*). *The joint size distribution is symmetric, that is*

$$p_{1,2}(S_1, S_2) = p_{1,2}(S_2, S_1). \tag{7}$$

The previous property is strictly related to the degree of symmetry in time of the underlying growth process. Roughly speaking, the idea is that if the "arrow of time" is reversed, the growth process of firms looks the same. For Markov processes, this property is usually known as "detailed balance" condition. Indeed this is the name used in Ref. [5]. Since we are interested in general results not depending on the specific nature of the process, for us Property 4 only implies that the probability for a firm to be of size S_1 at time 1 and of size S_2 at time 2, is the same as to be of size S_2 at time 1 and S_1 at time 2.