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Fokker–Planck type equations associated with fractional Brownian motion controlled by infinitely divisible processes

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HIGHLIGHTS

- The subordinated fractional Brownian motion with general inverse general subordinator is examined.
- We present the corresponding fractional Fokker–Planck type equation.
- We analyze main statistical properties of the considered system.
- In a special case we classify the examined process as accelerating-subdiffusion system.
- We present the simulation procedure.

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ABSTRACT

In this paper we study the anomalous diffusion process driven by fractional Brownian motion delayed by general infinitely divisible subordinator. We show the analyzed process is the stochastic representation of the Fokker–Planck type equation that describes the probability density function of an introduced model. Moreover, we study main characteristics of the examined process, the first two moments, that allow us in special cases for classification of it as a system with accelerating-subdiffusion property.

1. Introduction

The popular term anomalous diffusion refers to a phenomenon when the underlying object moves slower or faster than linearly in time. We distinguish two cases. The first one corresponds to subdiffusion when the centered second moment of the anomalous process $\{X(t)\}$ scales as $\mathrm{Var}\, X(t) \propto t^{\alpha}$, where $\alpha < 1$. While the other one is superdiffusion, namely $\mathrm{Var}\, X(t) \propto t^{\alpha}$, where $\alpha > 1$. Subdiffusion was observed in variety of systems, to name only few, biology: rat basophilic leukemia cells [1], neural cells [2], physics: charge transport in amorphous semiconductors [3,4] and even in finance [5]. In turn for superdiffusion, one is referred for example in context to physics [6], astrophysics [7], and biology [8]. For a landmark review of both these phenomena see Ref. [9].

One can name two approaches to tackle the subdiffusion problem. The first one is the fractional Brownian motion (FBM) $\{B_H(t)\}$ [10–12]. FBM of Hurst exponent H, where $0 < H < 1$, is a zero-mean Gaussian process with second moment $E(B_H^2(t)) = t^{2H}$. It is easy to observe that when $H < 1/2$ we obtain subdiffusive dynamics. The case $H > 1/2$ corresponds to superdiffusion and for $H = \frac{1}{2}$ FBM reduces to ordinary Brownian motion $\{B(t)\}$. The FBM is the most classical process

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commonly considered as a system with long-range dependence (called also long-memory). Let us mention for short-range dependent processes, the relation between values at different times decreases rapidly as the time difference increases. For processes with long-range dependence the relation is much stronger, [13]. The fractional Brownian motion has found many practical applications. We mention here finance [14], queuing systems [15], biology [16] and physics [17].

The second approach follows from the framework of continuous time random walks (CTRWs) with power law $t^{-1-\alpha}$ waiting times. This heavy-tailed waiting times introduce the periods when the test particle stays constant and after each flat period it performs a jump whose length is given by some probability density function (PDF) with finite second moment. Usual description of this scenario is in fractional Fokker–Planck equations (FFPE) [9,18] which describe PDF of anomalously diffusing objects. However deeper analysis of this model revealed that in fact subdiffusion is a combination of two independent mechanisms. The first one is the standard diffusion represented by some Itô process $\{X(\tau)\}$ (called an external process), while the second one is the waiting-time distribution represented by the so-called inverse α-subordinator $\{S_\alpha(t)\}$ [19–22], therefore as a consequence the mentioned CTRW in limit is a subordinated process $Y(t) = X(S_\alpha(t))$.

Stochastic representation of FFPE obtained through subordination of certain Langevin equations is a crucial thing in analysis of many complex phenomena. However the inverse problem, namely identification of FFPE associated with some stochastic process is important as well. This problem has been an object of many papers [21,23,24].

Inspired by these considerations we introduce general infinitely divisible (ID) inverse subordinator $\{S_\Psi(t)\}$ as a time-change. This process will serve as a new "clock" of a system. We define $S_\Psi(t)$ as

$$S_\Psi(t) = \inf\{\tau : T_\Psi(\tau) > t\}, \tag{1}$$

where $\{T_\Psi(\tau)\}$ is the strictly increasing Lévy process (i.e. process of independent stationary increments) [25] with the Laplace transform given by

$$\langle \exp(-uT_\Psi(\tau))\rangle = \exp(-\tau\Psi(u)). \tag{2}$$

The so-called Laplace exponent $\Psi(\cdot)$ is given by $\Psi(u) = \int_0^\infty (1 - \mathrm{e}^{-ux})\nu(\mathrm{d}x)$, where $\nu(\cdot)$ is the Lévy measure satisfying $\int_0^\infty (1 \wedge x)\nu(\mathrm{d}x) < \infty$. Further we assume that $\nu(0, \infty) = \infty$, to exclude compound Poisson process. Moreover we assume that $\Psi(\cdot)$ is a complete Bernstein function, with additional property that $\Psi'(k) > 0$. We observe that for every jump of $T_\Psi(t)$ there is a corresponding flat period of $S_\Psi(t)$, which is distributed according to the ID law.

In Ref. [26] authors consider the special case of Laplace exponent $\Psi(\cdot)$ and special case of $\{X(\tau)\}$ process driven by Brownian motion. The authors proved that under special assumptions the process $Y(t) = X(S_\Psi(t))$ has the so called accelerating-subdiffusion property i.e. it assumes two different short- and long-time scalings of mean square displacement (MSD) defined as a central second moment of given process. Let us mention that the accelerating-subdiffusion property is an extension of the pure subdiffusion. Processes with accelerating-subdiffusion find many practical applications like diffusion of telomeres in the nucleus of mammalian cells, molecules diffusing in living cells and a random motion of bright points associated with magnetic fields at the solar photosphere [27–29]. See also Refs. [30,31]. We extend the results of Ref. [26] in two fields. On one hand we consider a more general class of infinitely divisible inverse subordinators $S_\Psi(t)$ with general Laplace exponent for which the special case is the subordinator considered in Ref. [26]. On the other hand, as an external process we consider process $\{X_H(\tau)\}$ driven by fractional Brownian motion, that under some assumptions reduces to ordinary Brownian motion. We analyze the Fokker–Planck type equation (FPTE) for probability density function of the examined subordinated process $Y_H(t) = X_H(S_\Psi(t))$. Moreover, under additional assumptions we present main characteristics of the examined system such as moments. This allows for classification of $\{Y_H(t)\}$ as accelerating-subdiffusion process.

In the next section we introduce the general inverse subordinator examined also in Refs. [32–35]. In Section 3 we show the formula for FPTE for fractional Brownian motion delayed by general inverse subordinator. Then, in Section 4 we analyze moments of the examined process. The obtained formulas allow in special cases for classification of underlying system as process with accelerating-subdiffusion property. Next we present the simulation procedure in Section 5. Last section contains conclusions.

2. General ID subordinator

In this paper we consider the general infinitely divisible inverse subordinator $S_\Psi(t)$ defined in (1) for which the Laplace exponent $\Psi(\cdot)$ (see Eq. (2)) is a complete Bernstein function. In this case, according to Hausdorff–Bernstein–Widder theorem see Ref. [36], there exists a Lévy subordinator $T_\Psi(\tau)$ with Laplace transform given by formula (2). We associate the general subordinator $T_\Psi(\tau)$ with the following integro-differential-operator Φ_t, that for all functions $f \in C^1(\mathbb{R})$ is defined as

$$\Phi_t f(t) = \int_0^t M(t-y)\frac{\partial f(y)}{\partial y}\mathrm{d}y, \tag{3}$$

where the memory kernel $M(t)$ is defined via its Laplace transform in the following way:

$$\hat{M}(k) = \int_0^\infty \mathrm{e}^{-kt}M(t)\mathrm{d}t = \frac{\Psi(k)}{k}. \tag{4}$$