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Persistence intervals of fractals

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HIGHLIGHTS

- We investigate the relation between persistence intervals and fractal-like behavior.
- We show that for a certain class of objects this relation is straightforward.
- We argue that our analysis is useful in cases when fractality is a hidden feature.

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ABSTRACT

Objects and structures presenting fractal like behavior are abundant in the world surrounding us. Fractal theory provides a great deal of tools for the analysis of the scaling properties of these objects. We would like to contribute to the field by analyzing and applying a particular case of the theory behind the *P.H. dimension*, a concept introduced by MacPherson and Schweinhart, to seek an intuitive explanation for the relation of this dimension and the fractality of certain objects. The approach is based on recently elaborated computational topology methods and it proves to be very useful for investigating scaling hidden in dimensions lower than the "native" dimension in which the investigated object is embedded. We demonstrate the applicability of the method with two examples: the Sierpinski gasket – a traditional fractal – and a two dimensional object composed of short segments arranged according to a circular structure.

1. Introduction

Fractals surround us. They are present in nature [1], in the structure of our society [2,3], in our economic systems [4,5] and in all kinds of technologies we deal with day-by-day [6–8]. Their common property is their self-similar nature. Investigating them on different scales leads to the same conclusions.

One way to characterize these fractal structures is through the calculation of their dimension. For regular, exactly self-similar fractals this can be done analytically and there are various ways to estimate the dimension numerically for less regular cases. Perhaps the most wide-spread numerical method is the calculation of the Minkowski–Bouligand or box-count dimension [9]. Felix Hausdorff gave a rigorous definition of the dimension of an object [10] and it has been observed that these different definitions yield the same values, especially for fractals satisfying the open set condition [11]. By the dimension of an object, of course, we mean the "fractal dimension" and not the dimension of the space in which the object is embedded.

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Fractals are complex structures, their investigation boomed with the development of high-speed computers and, in fact, the analysis of many of their aspects would be almost impossible without the aid of these machines. As computational power is becoming less and less an issue, methods which in the past seemed unfeasible might actually revolutionize research in different fields. Since storing large topological complexes in the memory of the computers is not a problem anymore, some of these methods are offered by computational topology.

In this paper we intend to take a look at the relation between the dimension of fractal-like objects and their topological fingerprints encoded as persistence intervals of Betti numbers, a concept developed in computational topology [12]. The general theory behind this relation was recently laid down by MacPherson and Schweinhart [13].

2. Persistence intervals

Topology can be investigated by calculating so-called topological invariants. Topological invariants fix certain topological features of the topological space under investigation. The invariants we will focus on are the number of connected components, the number of holes and the number of voids in the investigated object. A connected component in this context means parts which in some way are connected to each other. For example, a regular ball has a single connected component (since it is a single piece), no holes and a single void (inside). A piece of paper also has a single connected component, no holes and no voids. If we would tear the paper in two, the system composed from the two (now separated) pieces of paper would have two connected components (since the two pieces are now disconnected from each other), no holes and no voids. If we would take a pencil and would poke a hole in one of the papers we would end up with a system with two connected components, one hole and no void. In the field of topology the mentioned topological invariants correspond to the so-called *Betti numbers* [14]. The number of connected components is the 0th Betti number, the number of holes is the 1st Betti number while the number of voids is the 2nd Betti number. This enumeration, of course, can be extended with the number of 4, 5, 6, etc. dimensional holes in higher dimensional spaces corresponding to 3rd, 4th, 5th, etc. Betti number.

2.1. A barcode representation of topology

Since we want to look at fractality, we are interested how the topology of the object is defined on different scales. Therefore we resort to the following abstraction: we will look to the number of holes on different scales. In order to achieve this, we will think about our objects as a dense (if it is the case) set of points and we adopt the following procedure:

- we calculate the Delaunay triangulation of the points
- we define a distance scale ε
- we connect all point pairs that are connected by an edge in the triangulation and are closer than ε
- we calculate the Betti numbers for the obtained structure.

We repeat these steps for all possible values of ε up to a maximal value $\varepsilon_{\max}$ and we "record" the formed connected components, holes and voids for each value of ε. $\varepsilon_{\max}$ may be defined as the longest edge in the triangulation. This construction procedure is analogous with the building of the so-called "alpha-complexes" [15].

At this point it is a valid question what is counted as a hole and how can voids form when all we do is connecting points with lines. The answer lies in the definition of topological building blocks which are points, lines, triangles, tetrahedrons and their higher dimensional analogs. According to this, triangles do not count as holes, instead they constitute faces. Similarly, the space enclosed by a tetrahedron is not counted as void.

For understanding the definition of a hole, imagine again a ball. If we perforate the membrane of the ball, in theory we could stretch the membrane out to a sheet, therefore a ball with a perforation is homeomorphic to a plane, that is, a single hole on a surface of a ball is in fact not a hole. If we perforate the ball again, this object will be homeomorphic with a plane with a hole on it. One needs to take into account these effects when counting holes.

After we scan the system with the procedure described above we know the numbers of connected components, holes and voids for each value of ε. The acquired information can be summarized in diagrams in the following way:

- we prepare different diagrams for connected components, holes and voids
- each instance of connected component, hole and void will be represented by a bar on the corresponding diagram
- the start point of each bar will correspond to the value of ε at which the corresponding instance came into existence, denoted by ε_b ("birth-point")
- the end point of a bar will correspond to the value of ε at which the corresponding instance ceased to exist, denoted by ε_d ("death-point").

The bars for connected components are somewhat special as connected components unite as ε increases. This process can be viewed as one of the connected components embeds the other one. Accordingly, the bar of the embedded component will end at the point where the component was embedded while the bar of the embedder component will continue until the letter will be embedded in another component. The role of embedded and embedder is arbitrary. It is easy to see that one of the bars for connected components will persist even at the highest values of ε as there will always be at least one connected component, thus this bar can be neglected as it does not carry any information.

The diagram compiled in the previously described way will be a barcode-representation of the topology of the system in which each bar represents the interval $[\varepsilon_b, \varepsilon_b)$ of ε over which the corresponding topological feature persists (persistence