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The control gain region for synchronization in non-diffusively coupled complex networks

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HIGHLIGHTS

- Model and synchronization control law scheme for non-diffusively coupled networks.
- Complete feasible control gain region for synchronization in non-diffusively coupled networks.
- Finding that a network may be synchronized by both negative and positive feed back control simultaneously.
- Timed synchronized region for evaluation of synchronization in finite time.
- A graphical method to estimate control gain with the minimum synchronization time.

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ABSTRACT

The control gain region for synchronization of non-diffusively coupled networks was studied with respect to three conditions: synchronization, synchronization in finite time, and synchronization in the minimum time. Based on cancellation control methodology and master stability function formalism, we found that a complete feasible control gain region may be bounded, unbounded, empty or a union of several bounded and unbounded regions, with a similar shape to the synchronized region. An interesting possibility emerged that a network could be synchronized by both negative and positive feedback control simultaneously. By bridging synchronizability and synchronizing response speeds with a settling time index, we have developed timed synchronized region (TSR) as a substitute for the classical synchronized region to study finite time synchronization. As for the last condition, a graphical method was developed to estimate control gain with the minimum synchronization time (CGMST). Each condition has examples provided for illustration and verification.

1. Introduction

Synchronization in complex networks [1,2] has motivated researchers for some time to help reveal underlying relationships between emerging phenomenon [3–9] and network structure. Most existing models [1–17] incorporate diffusive coupling hypotheses to ensure global behavior of the population of nodes, in which nodes state difference $\boldsymbol{x}_j - \boldsymbol{x}_i$

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always exist in the dynamical equation of node i if node j links node i. However, these models cannot cover all the coupling possibilities in actual networks, especially those in which nodes influence others by their state variables directly rather than by their state differences. For example, in price networks, the price of product (A) directly adds into the cost of product (B), and in traffic networks, the number of vehicles at junction (A) directly runs into the number at the next junction (B). It is highly desirable to investigate synchronization in non-diffusively coupled networks because it has universal implications [1,18,19]. In the study of network synchronization, if the network cannot reach an auto-synchronization we need to input a control signal to drive the network synchronal [2,12–15,19,20]. Whenever we try a decentralized state feedback control paradigm in the form $\boldsymbol{u}_i = cd\boldsymbol{\Gamma}(\boldsymbol{x}_i - \boldsymbol{s})$ [2,12,13,15], the whole feasible control gain region remains implicit. Although, there exist plenty of criteria [1,2,10–13] to judge whether a randomly selected control gain is functional, the trial and error procedure is time-consuming. Hence, a method to find a complete feasible control gain region is important. What is more, the shape of the control gain region may lead to further information concerning the synchronizability of networks. Regarding the feasible control gain region to synchronize a network, there exists three questions: (1) What is the extent of the feasible control gain region? (2) What is the extent of the feasible control gain region for synchronization in finite time [17,18,21]? (3) Which gain values can synchronize the network in the minimum time? This paper attempts to answer these three questions for non-diffusively coupled networks.

With the help of master stability function formalism [10,11], we found that the complete region of feasible control gains may be bounded, unbounded, empty or a union of several bounded and unbounded regions, with the same shape as the synchronized region has [15,16]. The interesting feature of the control gain region revealed the possibility that a network could be synchronized by positive and negative feedback control simultaneously. An example is provided to illustrate this new, so far, unreported finding. After that we developed timed synchronized region (TSR) to calculate the control gain region for finite time synchronization, and a graphical method to estimate control gain with the minimum synchronizing time (CGMST).

The rest of the paper is organized as follows. In Section 2, we introduce a general non-diffusively coupled network model and its synchronization control paradigm that forms the basis for the main results of the paper. The next three sections deal with the three synchronization conditions respectively. The last section summarizes our conclusions with some discussion.

2. Synchronization control of non-diffusively coupled networks

In classical diffusively coupled networks, the nodes state difference $\boldsymbol{x}_j - \boldsymbol{x}_i$ is adopted to form the network model $\dot{\boldsymbol{x}}_i = f(\boldsymbol{x}_i) + \sum_{j=1, j\neq i}^{N} h_{ij}\boldsymbol{\Gamma}(\boldsymbol{x}_j - \boldsymbol{x}_i)$, $i = 1, 2, \ldots, N$. By defining $h_{ii} = -\sum_{j=1,j\neq i}^{N} h_{ij}$, the ubiquitous model $\dot{\boldsymbol{x}}_i = f(\boldsymbol{x}_i) + \sum_{j=1}^{N} h_{ij}\boldsymbol{\Gamma}\boldsymbol{x}_j$ is formulized, where, the node state vector $x_i \in \mathbf{R}^n$, $f(\cdot) : \mathbf{R}^n \to \mathbf{R}^n$ stands for smooth vector-valued function, $\Gamma \in \mathbf{R}^{n\times n}$ represents the inner coupling matrix, $H = (h_{ij})_{N\times N} \in \mathbf{R}^{N\times N}$ is the outer coupling matrix with its entry $h_{ij} = h_{ji}$ denoting coupling strength if there is a connection between node i and node $j(i \neq j)$, $h_{ij} = h_{ji} = 0$ otherwise. By contrast, we define $h_{ii} = 0$ in non-diffusively coupled network model $\dot{\boldsymbol{x}}_i = f(\boldsymbol{x}_i) + \sum_{j=1}^{N} h_{ij}\boldsymbol{\Gamma}\boldsymbol{x}_j$. It is clearly indicated that node dynamics are directly affected by state variables of other nodes rather than their difference $\boldsymbol{x}_j - \boldsymbol{x}_i$.

In the linearized non-diffusively coupled network model $\dot{\boldsymbol{\eta}}_i = Df(\boldsymbol{s})\boldsymbol{\eta}_i + \sum_{j=1}^{N} h_{ij}\boldsymbol{\Gamma}\boldsymbol{\eta}_j + \sum_{j=1}^{N} h_{ij}\boldsymbol{\Gamma}\boldsymbol{s}$ with respect to synchronous state $s \in \mathbf{R}^n$, i.e., the equilibrium point of the isolated node satisfying $\dot{s} = f(\boldsymbol{s})$, $\Sigma_{j=1}^{N} h_{ij}$ is a nonzero constant because $h_{ii} = 0$, where $\boldsymbol{\eta}_i = \boldsymbol{x}_i - \boldsymbol{s}$ represents the node synchronization error and $Df(\boldsymbol{s})$ stands for the Jacobian matrix of f on $\boldsymbol{s}$. The nonzero value of $\Sigma_{j=1}^{N} h_{ij}$ implies an important fact that the influence of coupling will not disappear when all nodes achieve synchrony, namely $\boldsymbol{\eta}_i = 0$ for every $i = 1, 2, \ldots, N$. Hence, the immediate result is that pinning control [12–14,20] is not applicable for non-diffusively coupled networks. The reason is that the differential of the error vector $\dot{\eta}_i$ approaches constant $\sum_{j=1}^{N} h_{ij}\boldsymbol{\Gamma}\boldsymbol{s}$ if node i is not controlled, and then the nonzero $\dot{\boldsymbol{\eta}}_i$ will make synchrony unstable. For the same reason, the universal decentralized control law $u_i = d\Gamma x_i - b\Gamma s$ is not applicable either due to the diverse $k_i = \Sigma_{j=1}^{N} h_{ij}$ for all i, where d and b are real control gains. Let us consider the cancellation control law [14,19,22], a generally used control methodology with $u_i = d\Gamma x_i - (d+k_i)\Gamma s = d\Gamma\eta_i - k_i\Gamma s$, then we have a controlled network $\dot{\boldsymbol{\eta}}_i = Df(\boldsymbol{s})\boldsymbol{\eta}_i + \sum_{j=1}^{N} h_{ij}\boldsymbol{\Gamma}\boldsymbol{\eta}_j + d\boldsymbol{\Gamma}\boldsymbol{\eta}_i$. It appears similar to the controlled diffusively coupled network with the universal control law $\boldsymbol{u}_i = d\boldsymbol{\Gamma}\boldsymbol{\eta}_i$ [12,13,15]. However, they are different because we have introduced $-k_i\boldsymbol{\Gamma}\boldsymbol{s}$ to every node here as opposed to $-k_i\boldsymbol{\Gamma}\boldsymbol{x}_i$ in diffusively coupled networks.

Using the well-established decentralized cancellation control law given above, we are able to investigate the feasible region of control gain d to realize controlled synchronization. According to the master stability function method [10,11], only when eigenvalues of the characteristic matrix $\boldsymbol{G} = \boldsymbol{H} + d\boldsymbol{I}_N$ are located in the synchronized region, where matrix $\boldsymbol{A}_\alpha = Df(\boldsymbol{s}) + \alpha\boldsymbol{\Gamma}$ is Hurwitz stable, can the network achieve asymptotical synchronization. Here, α is a real variable representing the variation of eigenvalues of the characteristic matrix. The synchronized region S is a region on the real axis that may be bounded, unbounded, empty or a union of several bounded and unbounded regions [15,16]. We formulate S as

$$S = \bigcup_{i=1}^{N_S} S_i = \bigcup_{i=1}^{N_S} (\alpha_{i1}, \alpha_{i2}), \tag{1}$$