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Explicit form of the first-passage-time density for accelerating subdiffusion

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HIGHLIGHTS

- We find the Laplace transform of the first passage time density for the accelerating distributed order anomalous diffusion in the case of constant bias.
- We provide an explicit form of the first passage time density for the accelerating distributed order anomalous diffusion in the case of linear bias.
- We compare our theoretical results with numerical simulations for the corresponding subordinated stochastic models.

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ABSTRACT

The first passage time (FPT) statistics play a central role in different fields. Apart of description of the underlying system state and its evolution it is essential to investigate the time at which this state reaches a certain area for the first time. In this paper we study the FPT problem for recently developed models of an accelerating subdiffusion which extends, incorporating various short and long time scalings of the mean square displacement (MSD), popular models of subdiffusion. Based on fractional-order differential equations we derive the Laplace transform of the FPT density function in the case of constant bias. For a force depending linearly on the position we provide a full analytical formula for the FPT density in terms of generalized Mittag-Leffler functions and the Hermite polynomials. We also provide a numerical evidence that FPT properties are strictly connected with MSD scalings of the accelerating subdiffusion.

1. Introduction

Anomalous diffusion processes [1,2] are phenomena receiving nowadays more and more interests in various fields of science. They are usually characterized by an anomalous dependence of the mean square displacement (MSD) $\langle [X(t)-X(0)]^2\rangle = \langle \Delta X(t)^2\rangle$, which deviates from the linear dependence $\langle \Delta X(t)^2\rangle \propto t$ being a characteristic feature for the normal diffusion. Here $X(t)$ is a time dependent stochastic process describing either the position or velocity and $X(0)$ is its initial value. An anomalous diffusion behavior is observed in many systems [3,4]. In the particular case of one homogeneous power-law time scaling dependence of the MSD i.e. $\langle \Delta X(t)^2\rangle \propto t^{\alpha}$, the cases $0<\alpha<1$ and $1<\alpha<2$ refer to subdiffusion and superdiffusion respectively. However such anomalous kinetics represent only the simplest scenario of complex physical phenomena. Many systems demonstrate, for example different scaling of time through time domain, which can be described as a crossover between different power-laws or even non-power-law behavior [5–8].

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Recently stochastic models of systems with an accelerating subdiffusion were investigated [9,10], where the MSD exhibited different behavior in two regimes of time $t \ll \theta$ and $\theta \ll t$

$$\left\langle \Delta X(t)^2 \right\rangle \propto \begin{cases} (t/\theta)^{\alpha_1} & \text{for } t \ll \theta \\ (t/\theta)^{\alpha_2} & \text{for } t \gg \theta, \end{cases} \tag{1}$$

where $0 < \alpha_1 < \alpha_2 \le 1$. Such scaling means that for earlier times MSD scales with exponent α_1 while in later times with α_2. It is worth to mention that one can consider a scenario with an arbitrary number of the accelerating scalings. Such a peculiar and complex behavior was observed in many experiments, to name only a few ones: diffusion of telomeres [11], diffusion of molecules in living cells [12,13] or even a random motion of the bright points associated with the magnetic field at the solar photosphere [14]. One should also mention at this point that the accelerating subdiffusion is only a particular case of a more general description of complex diffusive processes based on randomizing the time clock of the process using a suitably chosen process S_t inverse to a nonnegative infinitely divisible random process [15–18].

In this paper we investigate further the model of the accelerating diffusion considered in Ref. [19] (see also [20]) and derive its first passage time (FPT) properties. Let us recall that accelerating diffusion phenomena can be modeled by a stochastic process with the probability density function (PDF) $p(x,t)$ satisfying the n-tuple distributed order time fractional diffusion equation (DOFDE)

$$\frac{\partial p(x,t)}{\partial t} = \sum_{i=1}^{n} p_i \theta^{1-\alpha_i} {}_0\mathcal{D}_t^{1-\alpha_i} \Big[-\frac{\partial}{\partial x} F(x) + K \frac{\partial^2}{\partial x^2} \Big] p(x,t), \tag{2}$$

where $p_i > 0$ and $\sum_{i=1}^{n} p_i = 1$, $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_n \le 1$ and $F(x)$ is some external force. Since the accelerating subdiffusion phenomenon encompasses simple subdiffusion phenomena with one diffusion exponent α in the present work we extend results on FPT properties derived in Ref. [21] in the presence of only one time scaling. It is worth to mention that the investigation of FPT properties of various stochastic models plays a central role in various applications. For a recent account of FPT for anomalous diffusion processes see Refs. [22,23] and the references therein.

Similarly as in Ref. [21] we consider two important cases. Namely a constant force $F(x) = -\nu$ and a linear force $F(x) = -\omega x$ with positive ω. Additionally we also compare numerically results obtained for anomalous diffusion with one time scaling [21] with those that we got in distributed order case.

This article is structured as follows. In Section 2 we present known results on FPT in case of mono-fractional diffusion equation and also recent advances in the distributed order case. In Section 3 we set up the foundations of the problem and discuss the stochastic representation of Eq. (2) in the case of the constant and linear force. In Section 4 we derive the Laplace transform of the FPT density function for a constant force. In Section 5 we obtain the full FPT density function when the force depends linearly on the position. The last Section 6 concludes the paper.

2. FPT properties in semi-infinite domain

In this paper we consider the FPT scenario for a distributed order n-tuple fractional diffusion in the semi-infinite interval. Since the problem that we cope with can be considered as a boundary value problem for the density of the process $p(x,t)$ with the absorbing boundary we formulate the following condition

$$p(0,t) = 0, \tag{3}$$

and the initial condition that the particle starts its evolution at point $x = a > 0$, namely

$$p(x,0) = \delta(x-a). \tag{4}$$

Similar investigations were carried out earlier [21], where the authors considered the mono-fractional diffusion equation, the special case of DOFDE with $n = 1$. In such a framework they have provided Laplace transform of the FPT density function in the case of a constant bias in the form

$$\hat{f}(s) = \exp\left(a(\nu - \sqrt{\nu^2 + 4Ks^\alpha})/2K \right), \quad 0 < \alpha \le 1 \tag{5}$$

and they proved that if $\nu \ge 0$, it is a Laplace transform of valid probability distribution. For $\nu < 0$, it can be read from (5) that the FPT density is defective i.e. the probability of the first passage is less than 1. These results are in accordance with the well known property of the standard Brownian motion $W_t^\nu = \nu t + \sqrt{2K} B_t$, $B(0) = a > 0$ with a drift ν, which corresponds to $\alpha = 1$ in (5), see Ref. [24–26]. Then for W_t^ν we have

$$f(t) = \frac{a}{\sqrt{2\pi t^3}} \exp\left(-\frac{1}{2t}(a - \nu t)^2 \right). \tag{6}$$

In the case of a linear force ($F(x) = -\omega x$) the authors were able to provide analytical formula for the density of the FPT distribution

$$f(t) = \frac{\omega}{\sqrt{\pi} t^{\alpha-1}} \sum_{n=0}^{\infty} \frac{(-1)^n}{4^n n!} H_{2n+1}\left(\sqrt{\omega/2Ka} \right) E_{\alpha,\alpha}\left(-(2n+1)\omega t^\alpha \right), \tag{7}$$

where $H_n(\cdot)$ are Hermite polynomials (see Eq. (49)) and $E_{\alpha,\alpha}(\cdot)$ is a Mittag-Leffler function defined in Eq. (21).