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General solution of a fractional diffusion–advection equation for solar cosmic-ray transport

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HIGHLIGHTS

- We solve the fractional diffusion–advection equation for solar cosmic-ray transport.
- We give its general solution.
- A numerical analysis of this equation is made.
- We use hypergeometric distributions.

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ABSTRACT

In this effort we exactly solve the fractional diffusion–advection equation for solar cosmic-ray transport and give its *general solution* in terms of hypergeometric distributions. Numerical analysis of this equation shows that its solutions resemble power-laws.

1. Introduction

There is a considerable body of evidence, from data collected by spacecrafts like *Ulysses* and *Voyager 2*, indicating that the transport of energetic particles in the turbulent heliospheric medium is superdiffusive [1,2]. Considerable effort has been devoted in recent years to the development of superdiffusive models for the transport of electrons and protons in the heliosphere [3–5]. This kind of transport regime exhibits a power-law growth of the mean square displacement of the diffusing particles, $\langle \Delta x^2 \rangle \propto t^\alpha$, with $\alpha > 1$ (see, for instance, Ref. [6]). The special case $\alpha = 2$ is called ballistic transport. The limit case $\alpha \to 1$ corresponds to normal diffusion, described by the well-known Gaussian propagator. The energetic particles detected by the aforementioned probes are usually associated with violent solar events like solar flares. These particles diffuse in the solar wind, which is a turbulent environment than can be assumed statistically homogeneous at

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large enough distances from the sun [1]. This implies that the propagator $P(x, x', t, t')$, describing the probability of finding at the space time location (x, t) a particle that has been injected at (x', t'), depends solely on the differences $x - x'$ and $t - t'$. In the superdiffusive regime the propagator $P(x, x', t, t')$ is not Gaussian, and exhibits power-law tails. It arises as solution a non local diffusive process governed by an integral equation that can be recast under the guise of a diffusion equation where the well-known Laplacian term is replaced by a term involving fractional derivatives [7]. Diffusion equations with fractional derivatives have attracted considerable attention recently (see Refs. [8–12] and references therein) and have lots of potential applications [13,14]. In particular, the observed distributions of solar cosmic ray particles are often consistent with power-law tails, suggesting that a superdiffusive process is at work.

A proper understanding of the transport of energetic particles in space is a vital ingredient for the analysis of various important phenomena, such as the propagation of particles from the Sun to our planet or, more generally, the acceleration and transport of cosmic rays. The superdiffusion of particles in interplanetary turbulent environments is often modeled using asymptotic expressions for the pertinent non-Gaussian propagator, which have a limited range of validity. A first step towards a more accurate analytical treatment of this problem is to consider solutions of a fractional diffusion–advection equation describing the diffusion of particles emitted at a shock front that propagates at a constant upstream speed V_{sh} in the solar wind rest frame. The shock front is assumed to be planar, leading to an effectively one-dimensional problem. Each physical quantity depends only on the time t and on the spatial coordinate x measured along an axis perpendicular to the shock front.

There are many ways of evaluating fractional derivatives of either functions or distributions.

(1) Defining the derivative using ultradistributions à la Sebastiao e Silva, as done in Ref. [15].
(2) Using Caputo's fractional derivatives, Riemann–Liouville ones, etc.

In the present contribution we re-visit the fractional diffusion–advection equation previously studied in other papers (see Ref. [15] and references therein). *Why?* Because we were able to give it a simplified treatment, clearly accessible to the entire scientific community, unlike that of Ref. [15], that appeals to ultradistributions of exponential type, a very sophisticated topic, accessible only to people familiar with extremely complex mathematics.

2. Formulation of the problem

Consider the equation

$$\frac{\partial f}{\partial t} = \kappa \frac{\partial^\alpha f}{\partial |x|^\alpha} + a\frac{\partial f}{\partial x} + \delta(x), \tag{2.1}$$

where $t > 0$ and $f(x, t)$ is the distribution function for solar cosmic-rays transport. Here the fractional spatial derivative is defined as

$$\frac{\partial^\alpha f}{\partial |x|^\alpha} = \frac{1}{\pi}\sin\left(\frac{\pi\alpha}{2}\right)\Gamma(\alpha+1)\int_0^\infty \frac{f(x+\xi) - 2f(x) + f(x-\xi)}{\xi^{\alpha+1}}\,\mathrm{d}\xi. \tag{2.2}$$

To solve this equation we use the Green function governed by the equation:

$$\frac{\partial \mathcal{G}}{\partial t} = \kappa \frac{\partial^\alpha \mathcal{G}}{\partial |x|^\alpha} + \delta(x)\delta(t). \tag{2.3}$$

With this Green function, the solution of (2.1) can be expressed as

$$f(x, t) = \int_0^t \mathcal{G}(x + at', t')\,\mathrm{d}t'. \tag{2.4}$$

In this work we obtain the solutions of Eqs. (2.1) and (2.3) using distributions as main tools [16].

For our task we use, as a first step, t the Green function through the use of the Fourier Transform given by

$$\hat{\mathcal{G}}(k, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathcal{G}(x, t)\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}x, \tag{2.5}$$

from which we obtain for $\hat{\mathcal{G}}$:

$$\hat{\mathcal{G}}(k, t) = -\kappa|k|^\alpha \hat{\mathcal{G}}(k, t) + \frac{1}{2\pi}\delta(t), \tag{2.6}$$

whose solution is

$$\hat{\mathcal{G}}(k, t) = \frac{H(t)}{2\pi}\mathrm{e}^{-\kappa|k|^\alpha t}, \tag{2.7}$$

where $H(t)$ is Heaviside's step function.