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Fixed points and stability in the two-network frustrated Kuramoto model

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HIGHLIGHTS

- We model two networked groups internally synchronising and externally competing.
- Fixed points describe the two groups phase locked or one group in two fragments.
- We analytically solve for thresholds for loss of synchrony leading to fragmentation.
- We numerically solve for one group on a tree graph and the other on a random graph.
- Numerical and analytical results for thresholds agree across a range of couplings.

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ABSTRACT

We examine a modification of the Kuramoto model for phase oscillators coupled on a network. Here, two populations of oscillators are considered, each with different network topologies, internal and cross-network couplings and frequencies. Additionally, frustration parameters for the interactions of the cross-network phases are introduced. This may be regarded as a model of competing populations: internal to any one network *phase* synchronisation is a target state, while externally one or both populations seek to *frequency* synchronise to a phase in relation to the competitor. We conduct fixed point analyses for two regimes: one, where internal phase synchronisation occurs for each population with the potential for instability in the phase of one population in relation to the other; the second where one part of a population remains fixed in phase in relation to the other population, but where instability may occur within the first population leading to 'fragmentation'. We compare analytic results to numerical solutions for the system at various critical thresholds.

1. Introduction

Dynamical processes on networks remain an ongoing area of research across complex systems of vastly different manifestations: physical, chemical, biological and social/organisational. The Kuramoto model of phase oscillators [1] on a general network is the most paradigmatic of formulations allowing for a myriad of variations depending on the specific area of application; for reviews see Refs. [2–5]. Among these, two variations of the Kuramoto model have recently become of interest in the literature: firstly, the *multi-network* formulation [6–10], where the entities being modelled on the network nodes may fall into quite different but nonetheless interacting populations (for example distinct species of organisms) each with very different network characteristics; and secondly, the *frustrated* Kuramoto model, where phase shifts are introduced

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in the interaction between adjacent oscillators such that the local interaction is not phase but frequency synchronising [11–13]. In this paper we consider an intersection of these two variations — a dual-network model, with frustrations introduced in the cross-network interaction. Our interest in such a model stems from our adaptation of the concept of adversarial or competitive interaction in social systems where two (or more) populations share a rivalry (for example, firms for market share [14], political or religious parties for membership [15], or popular opinion [16]). Note that the Kuramoto model has been adapted to social/human systems for several contexts: rhythmic applause [17], opinion dynamics [18] and human–robot musical performance [19]. There are also applications of coupled oscillator models on networks in which clustered populations are of interest, such as in the work of Ref. [20]. Our application of the Kuramoto model is to the decision-making process because of its essential *cyclicity*: for example the Perceptual Cycle model of Neisser [21] is based on a flow of an entity perceiving external data, cognitively processing that data, making decisions about future actions, and undertaking them thereby depositing new data into the external environment serving as an input into other entities' process. In this respect then, the frustration in the cross-interaction may represent the aim of one group to be 'a step ahead' of the decision-making of competitors. Inspired by this, the variation of the Kuramoto model we propose, for which we solve for fixed points, is close in spirit to the two network models of Refs. [7–9] which also include frustrations — however we consider finite general networks as opposed to a coupling of two complete graphs.

For a general unweighted undirected network described by an adjacency matrix $\mathcal{A}$, the Kuramoto model is expressed by the coupled differential equations:

$$\dot{\theta}_i = \omega_i + \sigma \sum_{j=1}^{N} \mathcal{A}_{ij} \sin(\theta_j - \theta_i), \quad 1 \le i \le N \tag{1}$$

where θ_i is the phase angle for node i, ω_i is an intrinsic frequency associated with the node, and σ a coupling constant. The basic behaviour of this system has been well explored: for sufficiently high coupling, all phases may eventually synchronise and approximately phase lock: $\theta_i \approx \theta_j \; \forall i, j$ (there are exceptions though, see Ref. [22]).

For two-network systems the equations are doubled and an additional cross network interaction is introduced. Each network represents a distinct population, collection or organisation of entities described by the phase at the node. To distinguish between the two competing populations we shall refer to them (based on competitive decision making approaches) respectively as 'Blue' and 'Red', and the entities at nodes of the respective networks as agents. Analysing such a model for fixed points, Lyapunov stability/instability and chaos allows us to pose questions such as: How should agents be linked to each other? How quickly should linked agents respond to each other compared to responsiveness to changes in their specific competitor? How much diversity in frequency can be afforded by any one population in light of its competitive interactions? And how can the aim for internal phase synchronisation be balanced against that for frequency synchronisation (being 'ahead') of the competitor. We show how the model can be unpacked to answer such questions.

While there are analogues of our approach in network control theory with a master network controlling another network, also known as 'outer synchronisation' [23], in our case the two networks stand in a symmetric relationship to each other — each may be seen as seeking to control the other. A plot of individual θ_i as functions of time illustrates our focus. In Fig. 1 we show blue and red θ_i which represent our two types of entities. Different networks connect blue and red internally, with a whole other network for blue to red; different couplings apply for entities of the same group or between the groups. The details behind this example are not important. But we observe that most blue entities are locked slightly ahead of the majority of the red population, while two blue entities are detaching from and then reconnecting with their groups. When lines diverge they reach 2π difference from each other, namely they rejoin on the other side of the unit circle, and another two red entities completely detach from their group, rejoin, and so on.

It is important to note that our intent is *not* to seek prediction of how such network connected entities behave in general for low couplings. Such a regime would undermine control. We seek, rather, to determine thresholds before 'control' is lost, so that one population may achieve internal phase synchronisation and external frequency synchronisation with respect to their competitor — such as the main blue population in Fig. 1. Within this we seek conditions such that the threshold for the type of fragmentation seen in the low-running red entities in Fig. 1 may be avoided. We derive such thresholds from linearisation and classical fixed point conditions. Such thresholds may be seen as bounding a 'strategy' (for example, for competitive decision making) for one population in relation to the other.

In the next section we detail the model and then analyse its behaviour close to fixed points for the two populations locking internally and then for one population fragmenting into two parts. We then illustrate our results with numerical examples. The paper concludes with a summary and future directions of the work.

2. The frustrated two-network model: 'Blue vs Red'

Consider N Blue agents in an internally connected network given by an adjacency matrix $\mathcal{B}_{ij}$, $(i, j = 1, \ldots, N \in \mathcal{B})$, and M Red agents connected in a network given by the adjacency matrix $\mathcal{R}_{ij}$, $(i, j = N+1, \ldots, N+M \in \mathcal{R})$, namely matrices with entries one if i and j are connected by a link and zero otherwise. Associated with each Blue agent $i \in \mathcal{B}$, is a phase β_i giving the position in a limit cycle; similarly ρ_i is the position in the limit cycle of a Red agent $i \in \mathcal{R}$. The 'Blue vs Red model'