Physica A 447 (2016) 535–544

Contents lists available at ScienceDirect

Physica A

journal homepage: www.elsevier.com/locate/physa

Fractal surfaces from simple arithmetic operations

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HIGHLIGHTS

- A new and general method for constructing surfaces with fractal self-affine properties is presented.
- The method is based on generalized bitwise arithmetic on real numbers.
- All generalized bitwise operators working on a finite alphabet are constructed.
- Fractal surfaces are discovered already in most simple arithmetic operations.
- A roughness exponent is identified for these surfaces, larger values of the exponent leading to coarser surfaces.

ARTICLE INFO

Article history:
Received 19 October 2015
Available online 24 December 2015

Keywords:
Fractals
Self-affinity
Free energy landscapes
Cityscapes
Hierarchical deposition

ABSTRACT

Fractal surfaces ('patchwork quilts') are shown to arise under most general circumstances involving simple bitwise operations between real numbers. A theory is presented for all deterministic bitwise operations on a finite alphabet. It is shown that these models give rise to a roughness exponent H that shapes the resulting spatial patterns, larger values of the exponent leading to coarser surfaces.

1. Introduction

Fractal surfaces [1,2] are ubiquitously found in biological and physical systems at all scales, ranging from atoms to galaxies [3–6]. Mathematically, fractals arise from iterated function systems [7,8], strange attractors [9], critical phenomena [10], cellular automata [11,12], substitution systems [11,13] and any context where some hierarchical structure is present [14,15]. Because of these connections, fractals are also important in some recent approaches to nonequilibrium statistical mechanics of steady states [16–19].

Fractality is intimately connected to power laws, real-valued dimensions and the exponential growth of details as resulting from an increase in the resolution of a geometric object [20]. Although the rigorous definition of a fractal requires that the Hausdorff–Besicovitch dimension D_F is strictly greater than the Euclidean dimension D of the space in which the fractal object is embedded, there are cases in which surfaces are sufficiently broken at all length scales so as to deserve being named 'fractals' [1,20,21]. Most of these surfaces are random and governed by growth, adsorption and deposition processes on interfaces [20,22–28].

The multidisciplinary field of *disorderly* surface growth has experienced a rapid development [24]. However, fractal surfaces formed by *deterministic* processes and hierarchical rules are also interesting and can be useful, e.g., as models for cityscapes [20]. In this article, by regarding 'fractals' in the broad sense mentioned in the previous paragraph, we construct

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http://dx.doi.org/10.1016/j.physa.2015.12.028

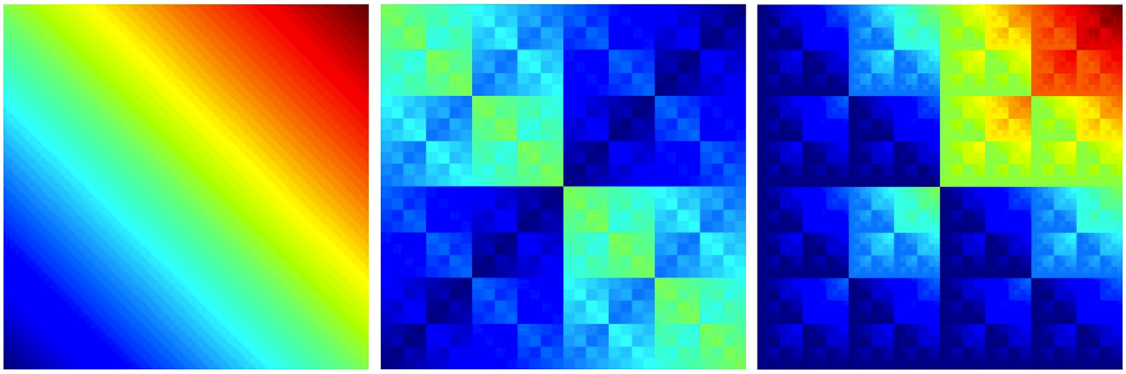

Fig. 1. The functions $f(x, y) = x+y$ (left), $g(x, y) = x+_2 y$ (center) and $h(x, y) = x\hat{+}_2 y$ (right) for $x \in [0, 1]$ and $y \in [0, 1]$. We have $f(x, y) = g(x, y)+h(x, y)$ everywhere. Dark blue means a zero value, green a value of 1 and dark red a value of 2. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

a wide variety of such surfaces and show that they possess fractal self-affine properties. If a surface $F(x, y)$ is self-affine it satisfies [22,23]

$$F(x, y) \sim b^{-H} F(bx, by) \tag{1}$$

where $H \in \mathbb{R}$ is the roughness exponent characterizing the self-affine scaling [1,29]. We prove that the surfaces here constructed obey Eq. (1). Although our construction proceeds abstractly, we illustrate it with specific numerical examples, and we believe that the generality of the method is such that it may find many applications in the modeling of complex physical systems. Although the notation may seem unfamiliar, the mathematics behind is elementary, and is based on generalized bitwise arithmetic on real numbers. To introduce this idea, let us consider two real numbers a and b (that we may truncate to a finite number of digits after the decimal point). If we expand a and b in base 10, we see that the ordinary sum of these numbers splits into the ordinary sum of two different parts

$$a + b = (a +_{10} b) + (a\hat{+}_{10} b) \tag{2}$$

where $+_{10}$ denotes addition modulo 10 and $\hat{+}_{10}$ denotes the contribution of the carries to the sum $a + b$. If $0 \leq a < 10$ and $0 \leq b < 10$ then, it is clear that $a + b = a +_{10} b$ if $a + b < 10$. If $a + b \geq 10$ then $a +_{10} b = a + b - 10$. If a and/or b are larger than 10, the sum $a + b$ is clearly reduced to separately considering the digits of a and b and adding them, taking care of the carries. Addition modulo 10 means that the carries are neglected. For example, if $a = 5.6782$ and $b = 3.6754$ we have that $a + b = 9.3536$. This sum can be seen as the ordinary addition of two different contributions $a +_{10} b = 8.2436$ and $a\hat{+}_{10} b = 1.1100$. The structure of the operators $+_{10}$ and $\hat{+}_{10}$ is very interesting. All positions within the numbers are independent of each other as regards the bitwise action of $+_{10}$: Each two digits corresponding to the same power of ten are added modulo ten and no carry is transferred from one position to another. Therefore, under the action of such operator, the positions within the number are 'uncoupled' and independent. Since each position within a number in a standard positional number system corresponds to a different power of the base, adding two real numbers modulo 10 means *performing the same operation at all scales, the latter being independent of each other*. We claim that, as a result of this, the function $g(x, y) = x +_{10} y$ (where x and y are real numbers) has fractal features. It is a discontinuous and non-differentiable function that exhibits the same details at all scales, displaying self-similarity. In fact this is generally the case for the function $g(x, y) = x +_p y$, where $p \in \mathbb{N}$, $p \geq 2$ is any base, with $+_p$ denoting *addition modulo p* [30,31]. *We shall prove below that such a function $g(x, y)$ obeys Eq.* (1). Furthermore, since $f(x, y) = x + y$ is not a fractal, the function $h(x, y) = x\hat{+}_p y = x + y - (x +_p y)$ must be a fractal as well so as to compensate the discontinuities of $g(x, y)$. In Fig. 1 the functions $f(x, y) = x + y$ (left), $g(x, y) = x +_2 y$ (right) and $h(x, y) = x\hat{+}_2 y$ are plotted for $x \in [0, 1]$ and $y \in [0, 1]$. While $f(x, y) = x + y$ is just an inclined plane, the functions $g(x, y) = x +_2 y$ and $h(x, y) = x\hat{+}_2 y$ exhibit a more complex behavior displaying 'squares within squares' at all scales. The behavior of these functions is such that under ordinary addition, all discontinuous jumps disappear at all scales, yielding the smooth function $f(x, y)$ (which is *everywhere* equal to $g(x, y) + h(x, y)$).

The purpose of this paper is to construct *all possible bitwise operators acting on a finite alphabet of p symbols* (to which $+_p$ and $\hat{+}_p$ do belong, and which we explicitly construct as well, as an example) and to elucidate their associated fractal features. We thus find a new kind of surfaces with self-affine properties, that we term 'patchwork quilts'. We constructively prove the general fact that bitwise operators locally acting on each point (x, y) in the plane $\mathbb{R}^2$ lead to these surfaces, providing explicit mathematical expressions for them. The results can be easily extended to hypersurfaces in $\mathbb{R}^n$. We explicitly construct the generalized bitwise operators involved in this process by making use of the framework of digital calculus, that we have very recently introduced [13,32,33]. We show that these abstract surfaces give rise to a roughness exponent H that shapes the resulting spatial patterns, larger values of the exponent leading to coarser surfaces.

Recently, we have found a new construction for fractals based on a fractal decomposition of a given function [33] linking the resulting fractal objects to finite group theory [33]. The construction of fractal surfaces given here is new and different