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Anomalous motion generated by the Coulomb friction in the Langevin equation

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Abstract

We study the stability of the Maxwell–Boltzmann (i.e., Gaussian) distribution for the density of states at equilibrium, against an arbitrary choice of the friction in the Langevin equation. We find that this distribution is Gaussian, if and only if the friction is Lipschitz continuous. In particular, we argue that the origin of the exponential (instead of Gaussian) velocity distribution (PDF) of particles when the viscous friction is replaced by the Coulomb friction in the Langevin equation with white noise is due to the non-Lipschitz continuity of the Coulomb friction, a feature of solid friction. The use of the Fokker–Planck equation to determine the exponential PDF is justified, since the subset on which the friction is not continuous is of zero probability ($v = 0$). The application to the motion of granular gases is discussed.

Keywords: Langevin equation; Solid friction; Granular materials

1. Introduction

The Langevin equation is the simplest stochastic differential equation, and extensively used in the framework of the Brownian motion which is a paradigm of the statistical physics. In absence of external force, this equation reduces to a first order stochastic differential equation in the velocity $\vec{v}$

$$\mathrm{d}\vec{v}/\mathrm{d}t = \vec{\mathscr{F}}(\vec{v}) + \vec{f}'(t) \tag{1}$$

with $\vec{\mathscr{F}}(\vec{v}) = -\eta\vec{v}/m$. m is the mass of the particle, the first term in the second member is the viscous friction, η is the friction coefficient proportional to the viscosity of the medium in which the particle moves, and $f(t)$ is the random thermal force which has the usual stochastic properties of being Gaussian with zero mean. In addition, we are interested here in the case where the noise $\vec{f}'(t) = \vec{f}/m$ is uncorrelated (white noise), i.e., its second momentum is a Dirac function

$$\langle \vec{f}' \rangle = 0, \quad \langle f'_\alpha(t) f'_\beta(t') \rangle = 2D\delta_{\alpha,\beta}\delta(t - t') \tag{2}$$

with D the diffusion coefficient (we shall return to this constant later on).

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In this paper, we address the following question: does this equation restore a Maxwell–Boltzmann distribution for the probability density of states in the long-time limit where a steady state is reached? The answer is non-trivial, and we find that it is yes if and only if F_{ext} has the particular property of being Lipschitz continuous. We also illustrate this purpose on two examples, the Lipschitz continuous case with the viscous friction, and the non-Lipschitz continuous case with the Coulomb friction.

It has been recently argued that the Coulomb friction is responsible for the fact that the probability density function (PDF) for the velocity in granular gases is exponential-like instead of Gaussian [1]. This result has been established on the ground of a simulation of granular particles in a quasi two-dimensional container under vertical vibration, taking into account the repulsion of the (monodisperse) contacted spheres, the rotation of the spheres and the Coulomb slip for tangential contact between the (monodisperse and spheric) particles, and random scatters fixed on the top board. This simulation shows that the Gaussian profile is restored if and only if the Coulomb forces are suppressed, so that the Coulomb friction is indeed responsible for the exponential profile in this simulation. The authors then proposed the Langevin equation with Coulomb friction instead of the viscous friction to describe the motion of this system in the horizontal plane, and the Fokker–Planck equation in this scheme gives an exponential-like stationary solution for the PDF [1]. On another hand, further analysis of this Langevin equation has been made [2] to explore the PDF for the velocity, including an effective gravity along one of the two dimensions. However, The PDF is found to be negative, at least in the small velocity part of the spectrum, which makes questionable the relevance of this equation to describe the motion in granular gases. We then found desirable to revisit the solution of the Langevin equation with the Coulomb friction, and the reasons why, in agreement with Ref. [1], the Coulomb friction is responsible for the outstanding form of the velocity PDF for the velocity in the horizontal plane for granular gases. We find that the non-Gaussian nature of the PDF is due to the fact that the Coulomb friction is ill-defined at $v = 0$, which is actually the characteristics of solid friction. The limits on the use of this equation and the associated Fokker–Planck equation to the case of granular gases in a tilted container are discussed.

2. General case: Lipschitz continuous force

The solution of Eq. (1) is known to exist in the case $\vec{\mathscr{F}}(\vec{v})$ is a Lipschitz-continuous function in the whole velocity space [3]. It means that there must exist a real constant K such that

$$\|\vec{\mathscr{F}}(\vec{v}) - \vec{\mathscr{F}}(\vec{v}')\| \leqslant K\|\vec{v} - \vec{v}'\| \tag{3}$$

for all $\vec{v}$ and $\vec{v}'$ in the whole range of times investigated. The Green function technique can be used to solve the problem in this case. Let $\vec{v}_{\mathrm{hom}}(t) = \vec{v}_0 Y(t)$ be the solution of the homogenous differential equation obtained by setting $\vec{\mathscr{F}}(\vec{v}) = 0$, with $\vec{v}_0 = \vec{v}(t = 0)$, meaning $Y(t = 0) = 1$. Then the causal Green function is $\vec{G}(t, t') = Y(t - t')\theta(t - t')$ so that the solution of Eq. (1) is simply

$$\vec{v}(t) = \vec{v}_{\mathrm{hom}}(t) + \int_0^t \mathrm{d}t' \, Y(t - t')\vec{f}'(t'), \tag{4}$$

where the integral is defined as the Itô stochastic integral of $Y(t)$ [4]. To be more specific, we can write this equation

$$\vec{w}(t) = \vec{v}(t) - \vec{v}_{\mathrm{hom}}(t) = \text{ms-}\lim_{n\to\infty} \sum_{i=1}^{n} Y(t_{i-1})[\vec{f}'(t_{i-1}) - \vec{f}'(t_i)]. \tag{5}$$

The t_i's are a partitioning of the time interval $[0 - t]$ $(0 \leqslant t_1 \leqslant t_2 \leqslant \cdots \leqslant t)$, and ms-lim means the mean square limit, i.e., the second member converges to the first member in the mean square sense. According to Eq. (5), $\vec{w}(t)$ is the sum of an infinite number of independent stochastic vectors. The central limit theorem then tells that $\vec{w}(t)$ has a Gaussian distribution of zero mean. Taking Eq. (2) into account, we readily find that the second moment is

$$\langle w_\alpha(t) w_\beta(t)\rangle = \delta_{\alpha,\beta} D'(t), \quad D'(t) = 2D \int_0^t \mathrm{d}t' \, Y^2(t - t'). \tag{6}$$