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Three related proposals for a theoretical definition of turbulence

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ABSTRACT

We pull together some developments in turbulence research to make three propositions: (1) when the steady-state solution of any transport equation produces multi-valued velocity fields, the system described is turbulent, (2) turbulent–laminar transitions are marked by the occurrence of a singularity in the derivative of velocity with respect to time, (3) the onset of turbulence in noble gases may be described quantum mechanically using the cell model of a gas, producing two testable laws describing the critical pressure of a turbulent gas.

1. Introduction

There is no unique, universally accepted theoretical definition of turbulence [1,2], despite the fact that turbulence is immediately recognizable from a large body of practical and heuristic measures documented in the literature [3,4]. Nevertheless, some criteria for the occurrence of turbulence have been suggested by David Ruelle [5], but unfortunately ignored by the large community of turbulence researchers.

Among the observations and suggestions by Ruelle are the following: (1) turbulence need not be defined by the Navier–Stokes equation alone, opening up the possibility of using other transport equations; (2) the onset of turbulence is probably marked by singularities, one of which may well be infinite gradients at a point in a fluid; and finally, (3) a system sensitively dependent on initial conditions is a candidate for a turbulent system.

Of the three points mentioned above, the last has been confirmed by modern, accurate pipe experiments [6–8]. In the last three experiments, it was shown that in a free efflux experiment through a pipe, the critical Reynolds number marking the turbulent–laminar transition depends on the starting pressure in an already turbulent gas. The critical Reynolds number at which turbulent flow becomes laminar is dependent on the initial pressure. We reproduce in this Letter the results of the free efflux experiment (courtesy of Physics Letters A, Elsevier) [6]. In the classical analysis from the continuum model of the Navier–Stokes equation, the plot of critical velocity against the ratio of initial pressure to the critical pressure, at which the turbulent–laminar transition occurs, should not be dependent on the initial pressure. Furthermore the plots show dependence on the species of the gas used in the experiment. In Fig. 1, all the experimental points should lie on one horizontal line, by virtue of the principle of classical scale invariance [9], which says that when the Navier–Stokes equation is rendered dimensionless, only the dimensionless Reynolds number is important, regardless of the constituent of the fluid under study. The species-dependence of the experiments in Refs. [7,8] is even more dramatic and poses a challenge to the traditional continuum model of turbulence.

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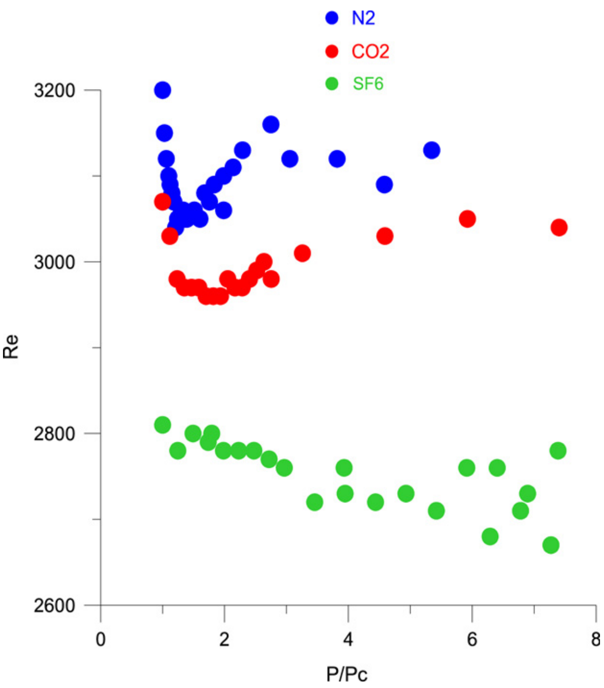

Fig. 1. The vertical axis is the critical Reynolds number. Using classical continuum analysis from the dimensionless Navier–Stokes equation, all points should collapse into one horizontal line and not be dependent on the ratio of initial pressure to the critical pressure. Furthermore the plots show dependence on the species of the gas used in the experiment.

2. New theoretical definition of turbulence

Let us start with Proposition I: A gas is turbulent when the steady-state solutions of the relevant transport equation produces a multi-valued velocity field. Each steady-state solution will be realized as a snapshot of the velocity field. Other snapshots can be produced by other allowed steady states. In time, the velocity field changes, as transitions occur from one allowed steady state solution to another.

We illustrate this with an example drawn from Ref. [10]. Consider the following transport equation for the velocity field $u(x, t)$ in one dimension:

$$\frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial u^2}{\partial x} - \sigma u = \frac{\sigma \Pi}{m}. \tag{1}$$

This equation represents the time evolution of a velocity field caricaturing a one dimensional system kicked by a quantum paddle which imparts an additional quantum of momentum Π to the particles constituting the gas, with a probability per unit time σ. We will consider a toroid geometry. Imagine a paddlewheel half- stuck into a doughnut-shaped vessel.

The steady-state solution is found by putting $\frac{\partial u}{\partial t} = 0$, giving the solution

$$u(x) = -\frac{\Pi}{m}\left[W_k\left(-\frac{m}{\sigma \Pi}\mathrm{e}^{-\frac{m\sigma}{\Pi}(x+C)-1}\right) + 1\right] \tag{2}$$

where C is a constant and Wk is the kth branch of the Lambert W function.

The function Wk is a solution of the equation $W \exp(W) = z$ in the complex plane [11]. It is multi-valued, making the stationary average velocity $u(x)$multi-valued.

With Ref. [10], we adopt a toroidal geometry, and put $x = L\sin(2\pi\theta)$, $0 \le \theta < 1$ ensuring a periodic boundary condition. The donut has a circumference. If $u(\theta) = 0$ then $C = -\Pi/(\sigma m)\ln(\Pi)$ to get

$$u_k(\theta) = -\frac{\Pi}{m}\left[W_k\left(-\frac{m}{\sigma}\mathrm{e}^{-\frac{m\sigma}{\Pi}(L\sin(2\pi\theta))-1}\right) + 1\right]. \tag{3}$$

We reproduce Fig. 1 from Ref. [10] for $k = 0, \pm 1, \pm 2, \pm 3$.

Each branch from Fig. 2 constitutes a possible snapshot of the velocity field. It is possible to think that thermal fluctuation triggers the different snapshots to evolve a time-dependent turbulent field. In particular, the zero branch immediately provides the possibility of velocity reversals, a signature characteristic of turbulence. We emphasize that what makes this solution possible is the introduction of the quantum of momentum.