Physica A 388 (2009) 407–418

Contents lists available at ScienceDirect

Physica A

journal homepage: www.elsevier.com/locate/physa

# Coupled continuous-time random walk approach to the Rachev–Rüschendorf model for financial data

Agnieszka Jurlewicz [a], Agnieszka Wyłomańska [a,*], Piotr Żebrowski [b]

[a] Hugo Steinhaus Center, Institute of Mathematics and Computer Science, Wrocław University of Technology, Wyb.Wyspiańskiego 27, 50–370 Wrocław, Poland
[b] Mathematical Institute, University of Wrocław, Pl.Grunwaldzki 2/4, 50–348 Wrocław, Poland

ARTICLE INFO

*Article history:*
Received 4 July 2008
Received in revised form 13 October 2008
Available online 5 November 2008

*PACS:*
89.65.Gh
05.40.Fb

*Keywords:*
Continuous-time random walk
Log-returns of financial instruments
Black–Scholes formula
Cox–Ross–Rubinstein model
Randomization
Alternative model

ABSTRACT

In this paper we expand the Rachev–Rüschendorf asset-pricing model introducing a coupled continuous-time-random-walk-(CTRW)-like form of the random number of price changes. Such a form results from the concept of the random clustering procedure (that resembles the coarse-graining methods of statistical physics) and, on the other hand, indicates applicability of the CTRW idea, widely used in physics to model anomalous diffusion, for describing financial markets. In the framework of the proposed model we derive the limiting distributions of log-returns and the corresponding pricing formulas for European call option. In order to illustrate the obtained theoretical results we present their fitting with several sets of financial data.



## 1. Introduction

In physics, the notion of continuous-time random walk[1] (CTRW for short) is commonly accepted as a powerful mathematical tool for analysis of relaxation and transport phenomena in complex systems that exhibit anomalous dynamical behavior resulting in a non-Gaussian asymptotic distribution of the diffusion front [1–14]. CTRW approach generalizes attempts to explain the observed departure from the normal (Gaussian) distribution based on the theory of Lévy-stable distributions. The alternative models obtained this way lead to scale mixtures of Lévy-stable laws (including the Gaussian law as a special case), where the mixing distribution depends on the detailed properties of the considered CTRW process [5,15,16]. For decoupled CTRW's[2] the mixing distribution is also stable and leads to the fractional stable limiting law [16–18]. Otherwise, more complicated mixing distributions may appear, connected not only with the stable but also with a generalized arcsine distribution (i.e. the beta distribution with parameters $p$ and $q = 1 - p$ for some $0 < p < 1$) [16,19,20].

Recently, CTRW's have been applied also in finance to model the evolution of log-prices, the risk process, or the forward rate dynamics [20–24]. Distributions of returns for real prices of financial instruments are the fundamental components in the portfolio security [25]. On the one hand, models of the capital market equilibrium study the structure of asset prices and

* Corresponding author.
*E-mail addresses:* Agnieszka.Jurlewicz@pwr.wroc.pl (A. Jurlewicz), Agnieszka.Wylomanska@pwr.wroc.pl (A. Wyłomańska), piotr.zebrowski@math.uni.wroc.pl (P. Żebrowski).

1 i.e. a walk with random waiting times between successive random jumps.

2 i.e. when the waiting times are independent of the jumps.

0378-4371/$ – see front matter © 2008 Elsevier B.V. All rights reserved.
doi:10.1016/j.physa.2008.10.041

its connection with empirical anomalies. On the other hand, option pricing models show empirical biases as the strike price bias or the smile effect, see Ref. [26].

Options have been traded for centuries but they remained relatively obscure financial instruments until the introduction of a listed option exchange in 1973. Since then option trading has enjoyed an expansion unprecedented on the US security markets. The modern option pricing theory begins in 1973. At that time F. Black and M. Scholes presented the first completely satisfactory equilibrium option pricing model taking into consideration the real aspect of financial markets [27]. In the same year, R. Merton extended their model using random interest rates [28], see also Refs. [29,30]. These path-breaking articles have formed the basis for many subsequent academic studies.

Option pricing theory is relevant to almost every area of finance. For example, virtually all corporate securities can be interpreted as portfolios of put and call options on the assets of the firm. Indeed, the theory applies to a very general class of economic problems; namely, the valuation of contracts where the outcome to each party depends on a quantifiable uncertain future event. Unfortunately, the mathematical tools employed in the Black–Scholes and Merton models are quite advanced and have tended to obscure the underlying economics [25,31].

The idea of looking at a binomial model as a discrete-time approximation to continuous-time diffusion were initially justified by J.C. Cox, J.E. Ross and M. Rubinstein in 1979 when they presented a simple discrete-time option pricing model (the CRR model, for short) [31,32]. This approach (categorized as a Lattice Model or Tree Model because of the graphical representation of the stock price over the large number of intervals or steps) includes the Black–Scholes model as a limiting case. The authors developed the proposed CRR model for a call option on a stock that pays no dividends. Also, they showed exactly how the model can be used to lock in pure arbitrage profits if the market price of an option differs from the value given by the model. By taking the limits in a different way, they obtained the Cox–Ross jump process model (1976) as another special case [33].

The Gaussian distribution of log-returns, appearing in the Black–Scholes model or as a limiting law in the CRR approach, is often observed being far from the empirical distributions of financial data [35]. Seeking more general and more realistic limiting models, in 1994 S.T. Rachev and L. Rüschendorf extended the CRR model by introducing two additional randomizations in the binomial price model: a randomization of the number of price changes and a randomization of the ups and downs in the price process [34,35]. As a result, they obtained the Rachev–Rüschendorff price models (the RR models, for short) with fat tails, higher peaks in the center and non-symmetric distribution, usually observed in typical asset return data [36].

In this paper, following the methodology proposed in Ref. [24], we expand the RR asset-pricing model introducing a coupled CTRW-like form for the random number of price changes. The proposed methods of randomization of the number of price jumps are based on the notion of random coarse graining transformation of CTRW's, introduced and studied in Refs. [5,16]; and they assume some agglutination of the successive price ups and downs in the groups (clusters) of random sizes. Such an approach provides quite realistic description of the asset price on the real market where indeed we observe randomizations of the value of successive groups of jumps. The considered form of the random number of price changes reveals significance of a new idea of CTRW's in the double-array limit scheme [18] in modeling of financial markets.

The article is structured as follows: Section 2 contains a brief survey on the idea and basic properties of the RR model. In Section 3 two clustering schemes for constructing the random number of price changes in the RR model (connected with different hedging strategies) are proposed. Then, in Section 4, the limiting distributions of log-returns and the corresponding pricing formulas for European call option (going beyond the classical Black–Scholes formula) are derived in the framework of the considered model. Finally, in Section 5 the third model, which combines properties of both previous schemes, is proposed and studied. Moreover, the obtained theoretical results are applied for analysis of several sets of financial data. Concluding remarks are given in the last section.

## 2. The Rachev–Rüschendorf model for financial market

The Rachev–Rüschendorf (RR) model is based on the classical CRR binomial model of market formed by a bank account $(B_k)_{k\geq 1}$ and some stock of value $(S_k)_{k\geq 1}$ [37]. In the CRR model one assumes that in the time interval $[0, T]$ for a fixed time $T$ (with 1 year as the time unit) both the value of bank account and the stock price change $n$ times at instants of time $k\frac{T}{n}$, $k = 1, \dots n$; and the evolution is given by the following rules:

$$B_k = \Lambda B_{k-1} = \Lambda^k B_0, \tag{2.1}$$

where $\Lambda = 1 + r\frac{T}{n}$ for the constant 1-year interest rate $r > 0$, while

$$S_k = u^{\epsilon_{n,k}} d^{1-\epsilon_{n,k}} S_{k-1} = S_0 \prod_{i=1}^{k} u^{\epsilon_{n,i}} d^{1-\epsilon_{n,i}}, \tag{2.2}$$

where for each $n$ the parameters $u = u_n > 1$ and $0 < d = d_n < 1$ correspond to jump up or down, respectively, of the stock price; and $(\epsilon_{n,k})_{k\geq 1}$ is a sequence of independent and identically distributed (i.i.d.) random variables such that $\Pr(\epsilon_{n,k} = 1) = p_n = 1 - \Pr(\epsilon_{n,k} = 0)$ for some probability $0 < p_n < 1$ of jumping up. The initial values $B_0$ and $S_0$ are positive constants.