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Growing network: Models following nonlinear preferential attachment rule

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HIGHLIGHTS

- We discuss a nonlinear preferential attachment rule for growing networks.
- For this rule the probability to attach to a node depends on degree k as $f(k)$.
- The node degree distribution can be calculated for any f based on derived formulas.
- The inverse problem of the calculation f for a given degree distribution is solved.
- So we can calibrate the graph model and explain growing networks features.

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ABSTRACT

We investigate the preferential attachment graphs proceeding from the following two assumptions. The first one: the probability that a new vertex connects to a vertex i is proportional to an arbitrary nonnegative function f of a vertex degree k. The second assumption: a new vertex can have a random number of edges. We derive formulas for any f to determine the vertex degree distribution $\{Q_k\}$ in generated graphs. The inverse problem is solved: we have obtained formulas, that allow from a given distribution $\{Q_k\}$ to determine f (the problem of a model calibration). The formulas allowing for any f to calculate the joint distribution of vertex degrees at the ends of randomly selected edge are also obtained. Some other results are presented in the paper.

1. Introduction

In the articles [1–3] it is shown that the classical Erdos–Renyi random graph [4] is not an appropriate model for such networks as WWW, scientific collaboration networks, electric power grids and others. This problem gave birth to a new branch of knowledge which is called a Network science [5,6], where new models for random networks are offered and the mutual correspondence between networks and models according to the structural characteristics is investigated. So far it was possible to construct a set of models that adequately reflect the basic properties and characteristics of real networks such as node degree distribution, the "small-world" effect, the diameter and the clustering coefficient [7,8] and etc. One of the main achievements is that the structural properties of a wide class of real networks could be explained by their development according to the simple general rule—the preferential attachment rule [9]. In accordance with the rule a network grows by adding new nodes, that are connected to the existing nodes, preferring those with higher degree of connectivity k (the "rich-get-richer" phenomenon [10]). In other versions of preferential attachment the nodes connectivity depends on their fitness to compete for links [11,12], and the time-dependence of preferential attachment can be considered [13], or other

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aspects affecting the preference [7,8]. The options of the preferential attachment rule, taking into account the nodes degree k, depend linearly on the nodes degree, or follow power law [14,15].

Nowadays to study large network structures such as social networks, web networks and the Internet [10,16] one model is widely spread, named as A. Barabasi and R. Albert random graph (BA graph) [9]. The BA graph starts with a small "seed" graph and grows by subsequent addition of new vertices with its $m = const$ incident edges. A new vertex with its incident edges that is added to the graph will be named a graph differential. The free ends of the graph differential edges attach to any randomly selected vertex of the graph. In the BA graph the probability p_i of attaching to a vertex i depends on the local vertex degree k_i:

$$p_i = \frac{k_i}{\sum_j k_j}. \tag{1}$$

The infinite addition of new graph differentials will result in growing of the infinite BA graph, which is called scale-free, because its vertex degree distribution is asymptotically power-law, i.e. the scale-free distribution. The exact expression for the stationary vertex degree distribution of the infinite BA graph is obtained in the articles [17]:

$$Q_k = \frac{2m(m+1)}{k(k+1)(k+2)}, \quad k = m, m+1, \ldots, \tag{2}$$

where Q_k is the probability that a randomly selected vertex has degree k.

From Eq. (2) it follows that $Q_k \sim 2m(m+1)k^{-3}$, and $Q_k \propto k^{-3}$ as $k \to \infty$. From the way of the BA graph construction it follows, that the mean vertex degree $\langle k \rangle$ is equal to $2m$. The equation $\langle k \rangle = 2m$ can also be derived from Eq. (2).

The asymptotic power-law vertex degree distribution of the BA graph is in a good agreement with the node degree distribution of many real networks, but not all of them [18,19]. But the general approach proposed by A. Barabasi and R. Albert helps to solve this problem. To solve it, we define and investigate a more general model of graphs—the graphs with nonlinear preferential attachment rule (the NPA graph).

In this paper we explore the preferential attachment rule in general, when the preferential attachment depends on the vertex degree k as an arbitrary nonnegative function $f(k)$, i.e. when the probability of connecting to a vertex i is determined in accordance with $p_i = f(k_i)/\Sigma_j f(k_j)$. The options are considered in cases when each added vertex has the same predetermined number of edges, and when each added vertex has the random number of edges. We investigate the case when each added vertex has the constant number of edges, and another case when each added vertex has x edges, where x is a random variable. We solve the following problems:

1. What form can take the vertex degree distribution in a graph that was grown using the preference function $f(k)$?
2. What is preference function $f(k)$ to choose so that the graph exactly corresponds to the predetermined vertex degree distribution?

   The solution of these problems is presented as a set of easy-to-use formulas. The solution of the second task allows us to grow the models which are precisely corresponding to the predetermined vertex degree distribution while modeling real networks.
3. What is the joint probability distribution for two degrees of vertices incident to randomly selected edge of the NPA graph?

Hence we obtain the asymptotic expressions for a graph clustering coefficient. By substitution the function $f(k) = k$ in the general formulas, we obtain the corresponding results for the BA graph, which are known or new (first found in Ref. [20]) results.

Used in the paper a rather simple mathematical apparatus has allowed us to obtain a number of other results that are given here. In particular, we offer a modification of the preferential attachment rule that allows you to control the clustering coefficient of a graph. Modeling real networks gives an opportunity to ensure the compliance the graph model of the networks and the corresponding networks for the vertex (node) degree distribution and the clustering.

The applied method to solve these problems can be considered, in a sense, as a addition and development of the method used in the paper of P. L. Krapivsky and S. Redner [14]. However, this method is a generalization, systematization and development of the method developed in our papers [20,21], and its core is in formulation and solution of the probability balance equations for those stationary probability distributions that we need for the problem.

2. Nonlinear preferential attachment rule

To grow the NPA graph we use a weight function $f(k)$ (weights): $f(k) > 0$, if $g \le k \le M$, otherwise $f(k) = 0$ (here $g \ge 1$, $M \le \infty$). The probability of attaching the graph differential edge (or arc) to a vertex i of the graph with N vertices is defined as:

$$p_i = \frac{f(k_i)}{\sum_j f(k_j)}, \quad i, j = 1, \ldots, N. \tag{3}$$