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Fokker–Planck equation with fractional coordinate derivatives

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ABSTRACT

Using the generalized Kolmogorov–Feller equation with long-range interaction, we obtain kinetic equations with fractional derivatives with respect to coordinates. The method of successive approximations, with averaging with respect to a fast variable, is used. The main assumption is that the correlation function of probability densities of particles to make a step has a power-law dependence. As a result, we obtain a Fokker–Planck equation with fractional coordinate derivative of order $1 < \alpha < 2$.

1. Introduction

In studying processes with fractal time and long-term memory a generalized kinetic equation was proposed in Ref. [1]. While the equation was of the master equation type, its main property was the presence of the power-type kernel for a probability density to make a step. This type of equation was compared with the Kolmogorov–Feller equation in Ref. [2]. In this paper, we would like to go farther and to show the conditions under which one can obtain a fractional generalization of the Fokker–Planck equation from the Kolmogorov–Feller equation.

Fractional calculus [3–5] has found many applications in recent studies in mechanics and physics, and the interest in fractional equations has been growing continually during the last years [6–12]. Fractional Fokker–Planck equations with coordinate and time derivatives of non-integer order has been suggested in Ref. [13]. The solutions and properties of these equations are described in Refs. [2,8]. The Fokker–Planck equation with fractional coordinate derivatives was also considered in Refs. [14–18].

The Kolmogorov–Feller equation is an integro-differential one, and it belongs to the type of master equations broadly used in different physical applications. It is well-known that the Kolmogorov–Feller equation can lead us to the Fokker–Planck equation [19], under some conditions. In this paper, we use the method of successive approximations with averaging with respect to a fast variable [20]. We suppose that the correlation function of probability densities, which are used in the Kolmogorov–Feller equation, is a power type. The Fokker–Planck equations with coordinate derivatives of order $1 < \alpha < 2$ are derived.

In Section 2, the Kolmogorov–Feller equation for the one-dimensional case is considered to fix notations and provide convenient references. We note that power-law probability to make a step gives the equation with a fractional derivative. In Section 3, we present a generalization of the Kolmogorov–Feller equation for the two-dimensional case. The method of successive approximations is used for this generalized equation in Section 4. In Section 5, we use averaging with respect to the fast variable, to derive fractional Fokker–Planck equations. Finally, a short conclusion is given in Section 6.

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2. Kolmogorov–Feller equation for one-dimensional case

2.1. Operator representation of the KF-equation

Let $P(t,x)$ be a probability density to find a particle at x at time instant t. The normalization condition for $P(t,x)$ is

$$\int_{-\infty}^{+\infty} \mathrm{d}x\, P(t,x) = 1 \quad (t > 0).$$

The Kolmogorov–Feller (KF) equation has the form

$$\frac{\partial P(t,x)}{\partial t} = \int_{-\infty}^{+\infty} \mathrm{d}x'\, w(x')[P(t,x-x') - P(t,x)], \qquad P(0,x) = \delta(x), \tag{1}$$

where $w(x')$ is the probability density of particle to make a step of the length x', and

$$\int_{-\infty}^{+\infty} \mathrm{d}x'\, w(x') = 1. \tag{2}$$

Let us introduce the operator representation of the KF-equation. We define the translation operator

$$T_{x'} = \exp\{-x'\partial_x\}, \tag{3}$$

such that

$$T_{x'}P(t,x) = P(t,x-x'), \tag{4}$$

and the finite difference operator

$$\Delta_{x'} = I - T_{x'}, \tag{5}$$

where I is an identity operator. Then the Kolmogorov–Feller equation (1) can be presented as

$$\frac{\partial P(t,x)}{\partial t} = L(\Delta)\, P(t,x). \tag{6}$$

Here we use the integro-differential operator

$$L(\Delta) = -\int_{-\infty}^{+\infty} \mathrm{d}x'\, w(x')\, \Delta_{x'}. \tag{7}$$

The operator (7) will be called the Kolmogorov–Feller operator. Note that power-law probability $w(x')$ in Eq. (1) allows us to introduce a derivative of non-integer order [4].

2.2. KF-equation with fractional coordinate derivative

The well-known fractional Caputo derivative [4] of order α is defined by

$${}^{C}D_x^{\alpha}P(x) = \frac{1}{\Gamma(1-\alpha)} \int_{-\infty}^{x} \frac{\mathrm{d}z}{(x-z)^{\alpha}} \frac{\partial P(z)}{\partial z}, \quad (0 < \alpha < 1). \tag{8}$$

The fractional Marchaud derivative [4] of order α is defined by

$$\mathbf{D}_x^{\alpha}P(x) = \frac{1}{\Gamma(-\alpha)} \int_{-\infty}^{x} [P(z) - P(x)] \frac{\mathrm{d}z}{(x-z)^{\alpha+1}}, \quad (0 < \alpha < 1). \tag{9}$$

Using $x' = x - z$, Eq. (9) has the form

$$\mathbf{D}_x^{\alpha}P(x) = \frac{1}{\Gamma(-\alpha)} \int_{0}^{\infty} \frac{\mathrm{d}x'}{(x')^{\alpha+1}} [P(x-x') - P(x)].$$

If the function $w(x')$ in the KF-equation (1) is the exponential function up to a small parameter ε such that

$$w(x') = \frac{a}{x'^{\alpha+1}} H(x') + O(\varepsilon), \tag{10}$$

where $H(x')$ is a Heaviside step function, then Eq. (1) can be presented as

$$\frac{\partial P(t,x)}{\partial t} = a\mathbf{D}_x^{\alpha}P(t,x) + O(\varepsilon), \qquad P(0,x) = \delta(x), \quad (0 < \alpha < 1). \tag{11}$$

This equation has a fractional coordinate derivative of order $0 < \alpha < 1$. Note that the function $w(x)$ is a probability density, and it should satisfy the normalization condition (2).