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The role of degree-weighted couplings in the synchronous onset of Kuramoto oscillator networks

Xiang Li

Department of Electronic Engineering, Fudan University, Handan Road 220, Shanghai 200433, PR China

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ABSTRACT

This paper investigates the role of asymmetrical degree-dependent weighted couplings in synchronization of a network of Kuramoto oscillators, where the conditions of coupling criticality for the onset of phase synchronization in degree-weighted complex networks are arrived at. The numerical simulations visualize that for networks having power-law or exponential degree distributions, asymmetrical degree-weighted couplings (with increasing weighting exponent β) increases the critical coupling to achieve the onset of phase synchronization in the networks.

1. Introduction

Collective synchronization in a large population of oscillators having natural different frequencies is a typical phenomenon in the fields of biology, physics, and engineering [1], whose mathematical descriptions are traced back to the pioneering work of Wiener [2] and Winfree [3]. Kuramoto refined this connection between nonlinear dynamics and statical physics, and formalized the solution to a network of globally coupled limit-cycle oscillators [4,1], answering the situation why the oscillators are completely de-synchronized until the coupling strength overcomes a criticality C_{syn}.

In the past decade, fruitful outcomes have witnessed the so-called small-world [5] and scale-free [6] phenomena of network connectivities in categories of large-scale complex networking systems including biological, engineering, social and economic systems [7–11], and, the discoveries of small-world and scale-free features in these natural and artificial complex networks have stimulated very wide concerns of how the complexity of a network structure facilitates and constrains the collective synchronous behaviors of a network, especially to the main interest of this paper, of Kuramoto oscillators [12–19].

For instance, Hong et al. reported their synchronization observations of a larger criticality of coupling strength on small-world networks than that of globally coupled networks [12]. And, it has also been stated the absence of critical coupling strength in frequency synchronization of a swarm of oscillators connected as a scale-free network having a power-law exponent $2 < \gamma \leq 3$ [13], which was further found to be determined by the largest eigenvalue of the adjacency matrix [14]. A more general investigation unveiled different paths to the emergent local patterns of synchronization [15], while Brede very recently proposed a method to generate a synchrony-optimized network of Kuramoto oscillators [16].

It should be noticed that a large body of previous investigations hold an assumption that every pair of connected nodes are coupled together with identical and symmetric couplings. However, in practice it is more general that pairs of nodes of a network are connected with non-identical and asymmetric couplings, where the collective synchronous behaviors of such networks are naturally very interesting to further explore [17–19]. It was visualized that with the asymmetric degree-weighted coupling C/k_i, where k_i is the degree of node i, a uniform coupling criticality of collective synchronization is independent of complexity of network topologies [19]. In this paper, we further study the role of a more general case of

E-mail address: lix@fudan.edu.cn.

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degree-weighted couplings C/k_i^β, where β is the degree-weighted exponent, in the synchronization onset of a network of Kuramoto oscillators.

2. The general case of Kuramoto oscillator networks with degree-weighted couplings

We consider a network of N coupled limit-cycle oscillators, whose phases θ_i, $i = 1, 2, \ldots, N$, evolve as

$$\frac{\mathrm{d}\theta_i}{\mathrm{d}t} = \omega_i + \sum_{j=1}^{N} C_{ij} a_{ij} \sin(\theta_j - \theta_i), \quad i = 1, \ldots, N, \tag{1}$$

where C_{ij} is the coupling strength between node (oscillator) i and node (oscillator) j, and a_{ij} is 1 (or 0) if node i is connected (or disconnected) with node j. Frequencies ω_i, $i = 1, 2, \ldots, N$, are randomly distributed following the given frequency distribution $g(\omega)$, which is assumed that $g(\omega) = g(-\omega)$.

Define the degree-weighted asymmetric coupling scheme for node i, $i = 1, 2, \ldots, N$

$$C_{ij} = C_i = \frac{C}{k_i^\beta}, \tag{2}$$

where the coupling strength C is a positive constant, β is the weighting exponent, and k_i is the degree of node i, which fits the given degree distribution $P(k)$ of a network. Therefore, we have

$$\frac{\mathrm{d}\theta_i}{\mathrm{d}t} = \omega_i + \frac{C}{k_i^\beta} \sum_{j=1}^{N} a_{ij} \sin(\theta_j - \theta_i), \quad i = 1, \ldots, N. \tag{3}$$

If for every node i, its degree $k_i = N$, $i = 1, 2, \ldots, N$, then when $\beta = 1$, model (3) equals the classic Kuramoto model for globally coupled networks [1,4].

Define the order parameter (r, Ψ) as Refs. [13,19]:

$$r\mathrm{e}^{\mathrm{i}\Psi} = \frac{\int \mathrm{d}\omega \int \mathrm{d}k \int \mathrm{d}\theta g(\omega) P(k) k \rho(k, \omega; t, \theta) \mathrm{e}^{\mathrm{i}\theta}}{\int \mathrm{d}k P(k) k}, \tag{4}$$

where $\rho(k, \omega; t, \theta)$ is the density of oscillators with phase θ at time t with the given frequency ω and degree k, which satisfies the normalization as

$$\int_0^{2\pi} \rho(k, \omega; t, \theta) \mathrm{d}\theta = 1. \tag{5}$$

Assume v to be the continuum limit of the right-hand side (r.h.s.) of Eq. (3), and each randomly selected edge is connected to the oscillator having degree k, frequency ω, and phase θ with probability $\frac{kP(k)g(\omega)\rho(k,\omega;t,\theta)}{\int \mathrm{d}kP(k)k}$. Therefore, determined by the continuity equation

$$\frac{\partial \rho}{\partial t} = -\frac{\partial(\rho v)}{\partial \theta}, \tag{6}$$

the density $\rho(k, \omega; t, \theta)$ evolves as

$$\frac{\partial \rho(k, \omega; t, \theta)}{\partial t} = -\frac{\partial}{\partial \theta}\left[\rho(k, \omega; t, \theta)\left(\omega + \frac{C}{k^\beta} k \frac{\int \mathrm{d}\omega' \int \mathrm{d}k' \int \mathrm{d}\theta' g(\omega') P(k') k' \rho(k', \omega'; t, \theta')}{\int \mathrm{d}k' P(k') k'} \sin(\theta' - \theta)\right)\right]. \tag{7}$$

Substituting Eq. (4) into Eq. (7) yields

$$\frac{\partial \rho(k, \omega; t, \theta)}{\partial t} = -\frac{\partial}{\partial \theta}\left\{\rho(k, \omega; t, \theta)\left[\omega + Ck^{1-\beta} r \sin(\Psi - \theta)\right]\right\}, \tag{8}$$

whose solution independent of time is

$$\frac{\partial}{\partial \theta}\left\{\rho(k, \omega; \theta)\left[\omega + Ck^{1-\beta} r \sin(\Psi - \theta)\right]\right\} = 0. \tag{9}$$

where $\rho(k, \omega; \theta)$ is assumed to be

$$\rho(k, \omega; \theta) = \begin{cases} \delta\left(\theta - \arcsin\left(\frac{\omega}{Ck^{1-\beta} r}\right)\right) & \text{if } \frac{|\omega|}{Ck^{1-\beta} r} \le 1 \\ \frac{D(k, \omega)}{|\omega - Ck^{1-\beta} r \sin\theta|} & \text{otherwise.} \end{cases} \tag{10}$$