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Self-organized criticality attributed to a central limit-like convergence effect

Wayne S. Kendal *

Division of Radiation Oncology, University of Ottawa, 501 Smyth Road, Ottawa, Ontario, Canada K1H 8L6

HIGHLIGHTS

- We explain self-organized criticality through a convergence effect related to the central limit theorem.
- This effect has as its focus of convergence the family of Tweedie exponential dispersion models.
- The Tweedie compound Poisson distribution inherently expresses both fluctuation scaling and $1/f$ noise.
- This compound Poisson distribution also can be used to predict the behavior of sandpile models.

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ABSTRACT

Self-organized criticality is a hypothesis used to explain the origin of $1/f$ noise and other scaling behaviors. Despite being proposed nearly 30 years ago, no consensus exists as to its exact definition or mathematical mechanism(s). Recently, a model for $1/f$ noise was proposed based on a family of statistical distributions known as the Tweedie exponential dispersion models. These distributions are characterized by an inherent scale invariance that manifests as a variance to mean power law, called fluctuation scaling; they also serve as foci of convergence in a limit theorem on independent and identically distributed distributions. Fluctuation scaling can be modeled by self-similar stochastic processes that relate the variance to mean power law to $1/f$ noise through their correlation structure. A hypothesis is proposed whereby the effects of self-organized criticality are mathematically modeled by the Tweedie distributions and their convergence behavior as applied to self-similar stochastic processes. Sandpile model fluctuations are shown to manifest $1/f$ noise, fluctuation scaling, and to conform to the Tweedie compound Poisson distribution. The Tweedie models and their convergence theorem allow for a mechanistic explanation of $1/f$ noise and fluctuation scaling in phenomena conventionally attributed to self-organized criticality, thus providing a paradigm shift in our understanding of these phenomena.

1. Introduction

Self-organized criticality (SOC) is a model proposed to explain the origin of $1/f$ noise and fractals in natural systems [1]. Bak, Tang and Wiesenfeld (BTW) postulated that irreversible dynamical systems with multiple spatial degrees of freedom could naturally evolve into self-organized and borderline unstable states without the fine-tuning of external parameters [1]. This borderline instability would predispose the system to fluctuations that would manifest as power law scaling. Temporally-based systems would therefore express $1/f^{\gamma}$ noise power spectra, with respect to the frequency f (in $1/f$ noise

* Tel.: +1 613 737 7700; fax: +1 613 247 3511.
E-mail address: wkendal@ottawahospital.on.ca.

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the power law exponent ranges between $0 < \gamma < 2$). Bak and Chen further proposed that spatial fractals could similarly manifest with fluctuations analogous to $1/f^\gamma$ noise [2].

Despite widespread interest no consensus has evolved towards a well-defined, mechanistic, explanation for SOC. BTW originally proposed a cellular automaton model known as the sandpile model [1]. It was later argued that this sandpile model really only expressed $1/f^2$ noise, indicative of Brownian motion [3]. Similar inconsistencies between the SOC hypothesis and observation have been found with real sandpiles [4]. Later simulation models like those of Maslov et al. [5], and other experimental sandpiles like those reported by Frette et al. [4] revealed true $1/f$ noise and thus provided substantiation of the BTW hypothesis. A nonconservative model for SOC was also proposed that yielded deterministic $1/f$ noise [6]. Further explanations for SOC have been derived from concepts like mean field theory [7] and information theory [8] but have similarly not achieved consensus as definitive explanations.

The logical construction of such theoretical models warrants consideration: if the premises on which a model is based are true then the results of the model would also seem to be true. In reality though, these conclusions would simply represent the choice of premises. Consider a sandpile simulation that yielded $1/f$ noise; it would do so because the premises were chosen to in order to produce $1/f$ noise, not necessarily because the model truly represented natural processes. In any theoretical model proposed to explain SOC (or $1/f$ noise) it would be important to ensure that the premises were both fundamental and representative of the range of natural processes where $1/f$ noise, or other related power law scaling, is observed.

Other difficulties with the SOC hypothesis stem from the qualitative nature of its original definition [1], and the ambiguity of terms like *self-organized* and *criticality* [7,9]. BTW, in their initial description, provided only a verbal description of how dynamical systems might possess a critical point as an attractor from which power law scaling could be expressed and reached in systems far from equilibrium [1]. A well-defined explanation for scaling phenomena like $1/f$ noise and related spatial fractals, though, would require precise mathematical description and unambiguous definitions.

There exists a second type of power law scaling, known as fluctuation scaling. It is characterized by a variance to mean power law that, similar to $1/f$ noise, has been observed within many natural systems [10]. Fluctuation scaling can be modeled by a family of statistical distributions known as the Tweedie exponential dispersion models (EDMs) [11], that inherently express this variance to mean power law [12]. Remarkably, the Tweedie models are also foci for a mathematical convergence effect, related to the central limit theorem (CLT), which thus explains this power law's wide manifestations [13]. The variance to mean power law in the context of self-similar stochastic processes also implies the existence of $1/f$ noise, and *vice versa* [14]. Consequently the Tweedie convergence theorem can be considered to underlie the genesis of $1/f$ noise. The question then arises as to how the Tweedie EDMs and their convergence effect might relate to SOC. Here the sandpile model [15], arguably the premier paradigm for SOC, is shown to express $1/f$ noise, fluctuation scaling, and to conform to the Tweedie compound Poisson distribution.

In the sections to follow a mathematical explanation for $1/f^\gamma$ noise and related power law behavior will be proposed based on the Tweedie EDMs, the Tweedie convergence theorem, and framed in the context of self-similar stochastic processes. This hypothesis (referred to here as the Tweedie hypothesis) does not necessarily presume an underlying dynamical process or critical point but rather is premised upon elementary closure properties of statistical distributions, a limit theorem on independent and identically distributed (i.i.d.) random variables and the correlation properties of self-similar processes. These premises are fundamental mathematically and are applicable to the range of natural processes where such scaling behavior is observed. The Tweedie hypothesis will be shown to be consistent with observations derived from ecology, genetics, physiology, cancer metastasis, random matrix theory and prime number theory, as well as with the sandpile model itself.

2. A sandpile model that demonstrates $1/f$ noise and fluctuation scaling

We begin with a review of the sandpile model as an example of $1/f$ noise and fluctuation scaling. Sandpile models are conventionally represented by a two-dimensional matrix, the elements $z_{\mu,\nu}$ which represent the number of sand grains that occupy discrete sites over a planar region. A grain of sand can be added to a random site and, if the number of grains exceeds a critical threshold, an avalanche will result with the grains being redistributed. Adjacent sites, if they attain the critical threshold by redistribution, generate further redistributions *et cetera* until the occupancy of all sites comes below threshold [1]. Any sand grains that cross regional boundaries are considered to be lost from the system. After the redistributions have run their course, the total number of grains in the system at that time t, $T(t) = \sum z_{\mu,\nu}(t)$, is recorded and the cycle of adding another grain of sand repeated [5]. A steady state will eventually evolve where the amount of sand being lost from the boundaries of the planar region more or less equals that being added. The variable $T(t)$ typically fluctuates about a mean value; the power spectrum of these fluctuations $S(f) = \left|\tilde{T}(f)\right|^2$ can be estimated from the Fourier transform $\tilde{T}(f)$.

Maslov's sandpile model was adopted here (see Section 2 of their paper, items i–iv, model 1A) [5]. Simulations were performed on a 2-dimensional $L_x \times L_y$ lattice ($L_x = 200$ $L_y = 2$) with the critical value $z_c = 1$. The boundary condition at $z(L_x + 1, y) = 0$ was kept open; the boundaries were closed on all other sides. If a lattice point exceeded the critical value, one grain of sand would be moved in the x direction and the other to the nearest neighbor along the y axis. Fluctuations of $Z(t)$ about the mean value $\overline{T(t)}$ were estimated $Z(t) = \left|T(t) - \overline{T(t)}\right|$ and analyzed. Once a steady state had been achieved, Z_i was assessed for the subsequent $N = 10^4$ time points.