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Renormalization group solution of the Chutes & Ladder model

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HIGHLIGHTS

- Model of biased persistent random walk with exact renormalization group solution.
- Non-universal exponent and de-localization transition.
- Pedagogical example of transport in a complex network.
- Strategy for foraging behavior or for "lifting" of Markov chains.

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ABSTRACT

We analyze a semi-infinite one-dimensional random walk process with a biased motion that is incremental in one direction and long-range in the other. On a network with a fixed hierarchy of long-range jumps, we find with exact renormalization group calculations that there is a dynamical transition between a localized adsorption phase and an anomalous diffusion phase in which the mean-square displacement exponent depends non-universally on the Bernoulli coin. We relate these results to similar findings of unconventional phase behavior in hierarchical networks.

1. Introduction

The variety of real networks found in biology, engineering, social sciences, and communication provides a need for new ideas to explore and classify the full range of critical phenomena that emerge as a result of the complex geometry [1–4]. Networks with a hierarchical organization of its sites have a long history in providing solvable models of statistical systems [5,6]. More recently, such networks, when turned hyperbolic with the addition of small-world links, have received considerable attention due to a variety of synthetic phase transitions that can be observed in such structures for well-known equilibrium models such as percolation [7–9] and Ising ferromagnets [10–14]. For instance, hyperbolic networks interwoven with geometric backbones provide solvable examples of discontinuous ("explosive") percolation transitions [15–17]. Previous studies of symmetric walks on such networks give simple examples of super-diffusion and shown close connections with Lévy flights [18,19].

Here, we consider a strongly biased variant of the familiar persistent random work [20,21]; an incrementally progressing walker in the forward direction undertakes back-jumps with a tunable frequency that is inversely related to the length of the long jump. But unlike those persistent walks that typically remain within the universality class of ordinary diffusion [21], the asymptotic behavior of our walks exhibit anomalous diffusion behavior with exponents that depend on the long-jump

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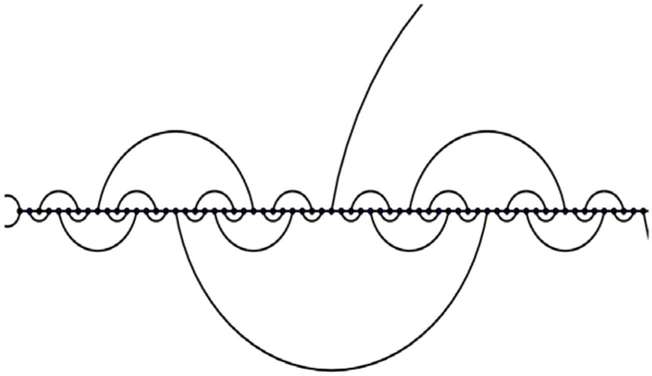

Fig. 1. Depiction of HN3 on a semi-infinite line. The leftmost site here is $n = 0$, which requires special treatment.

bias of the Bernoulli coin p. For instance, for the exponent that relates typical length and time-scales, $T \sim L^{d_w}$, such as determined through the mean-square displacement, we find

$$d_w(p) = \log_2\left(\frac{2-3p}{1-2p}\right), \tag{1}$$

which continuously ranges through $0 < p \leq \frac{1}{2}$, from straight ballistic motion, $d_w(0) = 1$, to all forms of anomalous super and sub-diffusion down to complete confinement or adsorption [22], $d_w = \infty$, for $p > \frac{1}{2}$. For instance, according to Eq. (1), exactly at $p = \frac{2}{5}$ the balance between step-by-step progression and occasional fall-backs results in an effectively diffusive spreading, $d_w = 2$, at long ranges. For smaller values of p, the chance to branch into a long back-jump is low and the progression dominates, while for larger p that progression ever more frequently gets stymied by occasional but significant set-backs.

Among other benefits, the tunability of the spreading walk provides strong control over transport properties [23,24] as well as mixing times. Combined with some forms of control, persistent walks such as this, for example, have received some renewed attention recently as a means to accelerate Markov chain algorithms via "lifting" [25–27]. Furthermore, quantized version of such walks are a fundamental ingredient in quantum algorithms [28–30].

First and foremost, our study here serves as a simple illustration of the unusual – and often non-universal – scaling behavior for dynamic processes on complex networks. In particular, hyperbolic networks such as the one considered here have been shown to possess a number of interesting, *synthetic* phase transitions in which the scaling behavior can be controlled by global parameters [11,31,32]. However, our model also provides a sense of what might happen in an ordinary, one-dimensional lattice with an incremental bias to walk one direction and back-jumps in the opposite direction, drawn randomly (annealed) from a Lévy-flight distribution [33]. This could be extended, for example, to model the behavior of directional transport of kinesin [34], interrupted with finite failure rate that leads to dissociation off the actin filament to reset the process. Certain forms of foraging behavior have also been described in these terms [35–38].

Our discussion is organized as follows. In the next section, we describe the network we are using and the random walk on it. In Section 4, we describe the results of our numerical simulations. In Section 5 we discuss our renormalization group calculation. Finally, we conclude our discussion in Section 6.

2. Network design

The network we are discussing in this paper [39] consists of a simple geometric backbone, a one-dimensional line of $N = 2^l$ sites $\left(0 \leq n \leq 2^l,\ l \to \infty\right)$. Each site on the one-dimensional lattice backbone is connected to its nearest neighbor. To generate the small-world hierarchy in these graphs, consider parameterizing any integer n (except for zero) *uniquely* in terms of two other integers (i, j), $i \geq 0$,

$$n = 2^i (2j + 1). \tag{2}$$

Here, i denotes the level in the hierarchy whereas j labels consecutive sites within each hierarchy. For instance, $i = 0$ refers to all odd integers, $i = 1$ to all integers once divisible by 2 (i.e., 2, 6, 10, . . .), and so on. In these networks, aside from the backbone, each site is also connected with (one or both) of its nearest neighbors *within* the hierarchy. We obtain the 3-regular network HN3 by connecting first all nearest neighbors along the backbone, but in addition also 1–3, 5–7, 9–11, etc., for $i = 0$, next 2–6, 10–14, etc., for $i = 1$, and 4–12, 20–28, etc., for $i = 2$, and so on, as depicted in Fig. 1. The site with index zero, not being covered by Eq. (2), is clearly a special place on the boundary of the HN3 that provides an impenetrable wall and ensures that the walks remains semi-infinite.