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# A two-dimensional non-Markovian random walk leading to anomalous diffusion

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## HIGHLIGHTS

- We propose a non-Markovian random walk model in 2D.
- The model is analytically solved for the first two moments.
- Several anomalous regimes are exhibited ($0 <$ Hurst exponent $< 1$).
- The full phase diagram with all anomalous regimes is drawn.
- The model can easily be upgraded to higher dimensions.

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## ABSTRACT

Exact solutions are rare for non-Markovian random walk models even in 1D, and much more so in 2D. Here we propose a 2D genuinely non-Markovian random walk model with a very rich phase diagram, such that the motion in each dimension can belong to one of 3 categories: (i) subdiffusive, (ii) superdiffusive, or (iii) normally diffusive. The main advance reported here is a different method, and the consequent physical insight, for analytically solving this model. Simpler non-Markovian models, such as Levy walks, have been solved in 2D, but it is not clear if the method of solution could be made to work for more complicated models such as the one studied here. We also report the exact solutions for the first two moments of the random walk propagator, along with the complete phase diagram. The latter is surprisingly rich and admits diverse diffusion regimes. Finally we discuss these results in the context of theoretical underpinnings as well as possible applications.


## 1. Introduction

Diffusion processes have been widely studied since the pioneering work of Einstein [1], more than a 100 years ago, for understanding the classical Brownian motion phenomena (see Refs. [2,3] and references therein). Anomalous diffusion [3–10], however, requires long range power law correlations in time for random walks for which the step sizes have a finite second moment. (When the second moment is allowed to be infinite, Lévy flights are possible [11].) Generalized Langevin equations [12,13], continuous time random walks [14,6] and the fractional Fokker–Planck equation [15,16] and the generalized master equation [17,18] are among the theoretical approaches traditionally used to study anomalous diffusion. Time

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correlations can be introduced in the model through memory effects. Such non-Markovian processes are important to many fields, such as physics, biology and socio-economy [19–25]. Long range memory correlations have been shown to give rise to anomalous diffusion, but exactly solvable non-Markovian walks are rare [26]. One distinguished non-Markovian model has become known as the elephant random walk model (ERW) [4], which introduces long range memory correlations by granting the walker full access to its whole history of past events. The one-dimensional ERW model presents anomalous diffusion and, furthermore, exhibits an elegant analytic solution.

Here we analyze a two-dimensional (2D) non-Markovian discrete time random walk model with exact solutions for the first two moments. The motivation for this study is to widen our understanding of random walks in more than one dimension, based on the fact that most walks in real life, consist essentially of two dimensional walks, like the motion of animals in general, including humans [27–29]. Solutions in 2D allow us to understand better the complex diffusive behavior associated with these models. In another direction, the probability distribution of the algebraic area of two dimensional discrete time random walk models, have been considered in the context of random magnetic impurities and the integer quantum Hall effect [30,31]. Moreover, many diffusive related phenomena are associated with long-range correlations, in which the future behavior is dependent on past memories as, for example, in the study of gait dynamics variations [32,33]. The study of a two-dimensional walk, particularly with exact solutions, can also help us to acquire the necessary knowledge and experience to formulate more sophisticated models in two, and possibly, higher dimensions. Three-dimensional models are important, for example, to model the transport of material inside living cells in microbiology, which is known to be driven by subdiffusion [34–37], due probably by the cell's crowded environment [38–40]. However, attempts to tackle non-trivial non-Markovian models in higher dimensions, are always faced with great mathematical complexities, mostly due the long range correlations in memory that are inherently built in the dynamics of such models. By working in incremental steps we can establish a set of mathematical tools that can lead to exact solutions, understand their limitations and possibly overcome them. We hope this will bring us one step closer to formulate analytically treatable 3D models with strong memory correlations. The models we are dealing with have the great advantage of displaying several diffusion types by a conveniently simple change of the model's parameters. This feature can make them important allies in our attempt to understand the underlying microscopic mechanisms responsible for transport phenomena in the real world.

The 2D model we propose contains memory of all previous steps in the decision making process at present time. Since there are long-range temporal correlations (due to the memory), the model is non-Markovian. In other words, no $n$-step Markov process can, in principle, approximate the behavior of such a model, because Markov processes have short-range (e.g., exponentially decaying) correlations. The model uses stopping points along one direction to flip the motion into another direction, and is formulated having two other models in mind: the first is the elephant random walk model [4] published in 2004, and the second is its generalization by Kumar, Harbola and Lindenberg (KHL) [5] in 2010, by including stops (or pauses) in the dynamics. Both the ERW and the KHL model, represent one-dimensional (1D) walks with strong long-range memory correlations.

Analytic solutions for the first two moments are presented, allowing us to derive the Hurst exponents and uncover the model's diffusion behavior. The full phase diagram of the 2D model is exhibited, displaying all main types of diffusion regimes, namely, sub-diffusive, normal and superdiffusive. The regimes are also classified as escaping and non-escaping regimes, according to the asymptotic behavior of the first moment. The rich variety of phases endows the model with great flexibility, which is ideally suited for many uses, possibly even real applications. In fact, the two-dimensional walk opens up a vast possibility of applications, considering that many of the walks in the real world are essentially 2D walks. Although the model is formulated in two dimensions, its generalization to higher dimensions is straightforward.

## 2. The model

In this section we describe the details of the two-dimensional random walk model that will be discussed throughout the rest of the paper. We start with a walker currently at a position $(X_t, Y_t)$ at time $t$, seeking to move to $(X_{t+1}, Y_{t+1})$ at time $t+1$. Its new position can be written generally as $(X_{t+1}, Y_{t+1}) = (X_t, Y_t) + (\sigma_{t+1}, \tau_{t+1})$. The random variables $\sigma_{t+1}$ and $\tau_{t+1}$ are related to the walk along the $x$-direction and the $y$-direction, respectively, and represent a memory correlated stochastic noise. They are allowed to take on the values $0, \pm 1$, with the following meanings: $\sigma_{t+1} = +1$ ($\sigma_{t+1} = -1$) for a right (left) step along the $x$-direction and $\sigma_{t+1} = 0$ for a halting point. Likewise, $\tau_{t+1} = 0, \pm 1$, have the same meanings, but along the $y$-direction.

The basic stochastic equation is written in terms of four variables $p_x, q_x, p_y, q_y$, ranging from 0 to $+1$, related by

$$p_x + q_x + p_y + q_y = 1. \quad (1)$$

In this equation, $p_x + q_x$ ($p_y + q_y$) is the probability of attempting a walk along the $x$-direction ($y$-direction) as explained in Fig. A.1. Simultaneous steps, i.e., along both directions at a given time, are not allowed in this formulation. One starts with the selection of one of the four stochastic variables in Eq. (1) by choosing a random number between 0 and 1 from a uniform distribution. Having decided which direction to go, one has to set the value of $\sigma_{t+1}$ (for the $x$-direction) or $\tau_{t+1}$ (for the $y$-direction). In order to understand how this is done, let us suppose that the $x$-direction has been selected using Eq. (1). In this case, no step will be taken along the $y$-direction and $\tau_{t+1}$ is thus immediately set to zero. Furthermore, notice that, the selection of the $x$-direction using Eq. (1), implies that the selection of either $p_x$ or $q_x$ has already been done (see Fig. A.1).