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Codifference as a practical tool to measure interdependence

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HIGHLIGHTS

- The structure of dependence of different processes is examined.
- We present the general measure of dependence defined for all infinitely divisible processes, namely codifference.
- We show that the codifference for Gaussian processes reduces to classical covariance.
- We present the form of the codifference for processes with finite and infinite variance.
- We show how to estimate the codifference from real data.

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ABSTRACT

Correlation and spectral analysis represent the standard tools to study interdependence in statistical data. However, for the stochastic processes with heavy-tailed distributions such that the variance diverges, these tools are inadequate. The heavy-tailed processes are ubiquitous in nature and finance. We here discuss codifference as a convenient measure to study statistical interdependence, and we aim to give a short introductory review of its properties. By taking different known stochastic processes as generic examples, we present explicit formulas for their codifferences. We show that for the Gaussian processes codifference is equivalent to covariance. For processes with finite variance these two measures behave similarly with time. For the processes with infinite variance the covariance does not exist, however, the codifference is relevant. We demonstrate the practical importance of the codifference by extracting this function from simulated as well as real data taken from turbulent plasma of fusion device and financial market. We conclude that the codifference serves as a convenient practical tool to study interdependence for stochastic processes with both infinite and finite variances as well.

1. Introduction

Stochastic processes with diverging variance, like alpha-stable Lévy motion, or Lévy flights, are ubiquitous in nature and finance. This class of non-Gaussian random processes was first investigated by French mathematician Paul Pierre Lévy [1].

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Lévy stable probability laws are important, because due to the Generalized Central Limit Theorem they attract distributions of sums of random variables with diverging variance, similarly to the Gaussian law that attracts distributions with finite variance. In the preface to their classical monograph "Limit Distributions for Sums of Independent Random Variables" [2] Gnedenko and Kolmogorov give a big respect to the Lévy's discovery, saying that "... the theory of these laws and the general limit theorems connected with them will receive in time diverse applications". On the next page they continue: "... "normal" convergence to the non-normal stable laws ... undoubtedly must already be considered in any comprehensive text in, say, the field of statistical physics". Due to the Generalized Central Limit Theorem Lévy stable laws naturally appear when evolution of a system or result of an experiment is determined by a sum random factors.

Remarkably, the first appearance of non-Gaussian stable laws in physics was due to Holtzmark even before the works of Lévy [3]. Holtzmark showed that the probability distribution of local electric fields in a system of randomly distributed Coulomb centers is given by three-dimensional symmetric stable law. The same is true for the distribution of gravitational force acting on any given object from an infinite number of stars [4]. The Holtzmark distribution is now widely used in astrophysics and plasma physics, being a famous representative of the whole family of symmetric stable laws arising in spatially homogeneous many-particle systems with long-range interactions [5–9].

A fundamental property of the Lévy stable laws is the "heavy tails", i.e. asymptotic power-law form of the probability density functions decaying as $|x|^{-1-\alpha}$, where α is the index of stability, or the Lévy index, varying between 0 and 2. Hence, the variance diverges, and for that reason these laws appear naturally in the description of many fluctuation processes characterized by bursts or large outliers. Bursty fluctuations are inherent to many phenomena far from equilibrium, such as, e.g., turbulence and financial market dynamics. Indeed, stable distributions and stable processes have become regular tools in financial data analysis [10–12], and analogies and dissimilarities between price dynamics and hydrodynamic turbulence have been discussed in the literature [13,14]. In this context we also mention a good example of a paper that gives detailed comparison between stable and multiplicative models in the space plasma turbulence [15].

Lévy statistics may also appear asymptotically due to the Generalized Central Limit Theorem like, for example, in non-Brownian random walks with jumps and/or waiting times obeying heavy-tailed distributions, see the reviews [16–19]. Examples of Lévy flights in nature range from elementary process of diffusion of photons in hot atomic vapors [20], light propagation in fractal medium called Lévy glass [21] and a "paradoxical" particle diffusion on a fast-folding polymers [22,23] to circulation of dollar bills [24] and behavior of the marine vertebrates in response to patchy distribution of food resources [25], to name a few. The Lévy flight dynamics can also stem from a simple Brownian random walk in systems whose operational time typically grows superlinearly with physical time t [26,27]. Such dynamics was observed experimentally in fluorescence recovery after photobleaching [28]. Stably distributed random noises are observed in such diverse applications as plasma turbulence (density and electric field fluctuations) [29–32], stochastic climate dynamics [33,34], physiology (heartbeats [35]), electrical engineering [36], biology [37], and economics [38]. Lévy noise driven Langevin systems demonstrate quite unusual and fascinating properties, for example, Boltzmann-type equilibria are non-attainable; instead, the stationary states called "confined Lévy flights" may appear in steep potential wells [39–42].

Other prominent examples of the processes with heavy-tailed distributions is Lévy Ornstein–Uhlenbeck (OU) process describing the overdamped harmonic oscillator driven by alpha-stable Lévy noise, and fractional Lévy stable motion. The Lévy OU process is a natural generalization of the Gaussian OU process; such generalization has gained popularity, e.g., in finance [43–45]. The weakly damped harmonic oscillator driven by Lévy noise was discussed in Refs. [46,47]. The stochastic behavior of the Lévy Ornstein–Uhlenbeck capacitor system was investigated in Ref. [48], and the approach based on Poisson-superposition jump structure of Lévy motions was used to analyze Lévy driven OU dynamics [49]. The Lévy OU process accounts for interdependence (association) of exponential type. On the contrary, fractional motions and fractional noises have an infinite span of interdependence [50–52]. Fractional processes are also widely spread in applications [53–55]. Indeed, in a large class of many-particle systems whose overall dynamics is Markovian, the probe particle coupled with the rest of the system through space correlations exhibit fractional motion with long-ranged non-Markovian memory effects [56–58]. Fractional Lévy stable motion with long-range dependence was detected in beat-to-beat heart rate fluctuations [35], in solar flare time series [59], and was shown to be a model qualitatively mimicking self-organized criticality signatures in data [60].

What is the measure of interdependence for the processes with infinite variance? Apparently, correlation or spectral power analysis, strictly speaking, cannot be used. The alternative measures of dependence are rarely discussed in application-oriented literature.

The notion of covariance (CV) used in correlation analysis, can be generalized for the alpha-stable Lévy process, leading to the notion of covariation [51,61]. Some results related to the covariation of autoregressive process with Lévy stable distribution are presented in Ref. [62]. The other characteristics is the Lévy correlation cascade [63]. It is defined for infinitely divisible processes, and its properties are discussed in Ref. [64] in detail, see also Ref. [65]. Both the covariation and the Lévy correlation cascade are of limited practical value, because their definitions are based on the Lévy random measure which can hardly be restored from experimental data series.

Our paper deals with another measure of interdependence called codifference (CD). It is based on the characteristic function of a given process, therefore it can be used not only for alpha-stable processes. Moreover, the codifference in the Gaussian case reduces to the classical covariance, so it can be treated as the natural extension of the well-known measure. On the other hand, according to the definition, it is easy to evaluate the empirical codifference which is based on the empirical characteristic function of the analyzed data. It is worth to mention that the codifference is closely related to the so-called