Physica A 387 (2008) 6252–6262

Contents lists available at ScienceDirect

Physica A

journal homepage: www.elsevier.com/locate/physa

Geometrical minimum units of fracture patterns in two-dimensional space: Lattice and discrete Walsh functions

Yuta Nishiyama [a], Kazuyoshi Z. Nanjo [b], Kazuhito Yamasaki [a,*]

[a] Department of Earth and Planetary Sciences, Faculty of Science, Kobe University, Nada, Kobe 657-8501, Japan

[b] Swiss Seismological Service, ETH Hoenggerberg, HPP P, 8093 Zurich, Switzerland

ARTICLE INFO

Article history:
Received 19 December 2007
Received in revised form 9 April 2008
Available online 17 July 2008

PACS:
91.60.Ba
06.20.F-
89.70.Cf
02.10.Ab

Keywords:
Fracture
Symmetropy
Symmetry
Entropy
Lattice
Walsh function

ABSTRACT

We present the geometrical minimum units of fracture patterns in two-dimensional space. For this analysis, a new method is developed from the algebraic approach: the concept of lattice (a type of partially ordered set) is applied to the discrete Walsh functions that have been used to measure symmetropy (an object related to symmetry and entropy) of fracture patterns. We concluded that the minimum units of fracture patterns can be expressed as three kinds of lattice. Our model is applied to the temporal change of the spatial pattern of acoustic-emission events in a rock-fracture experiment. As a result, the symmetropy of lattice decreases with the evolution of fracture process. We find that the pre-nucleation process of fracture corresponds to the subcritical states, and the propagation process to the critical states. Moreover, using a particular mathematical structure called sheaf on a lattice, we suggest the algebraic interpretation of fracture process, and provide justification to regard fracturing as an irreversible process.

1. Introduction

The fracture phenomena are extremely common, but the spatial patterns of fractures or earthquakes seem to be too complicated for quantification. Therefore, several new concepts, such as fractal geometry and a self-organized criticality (SOC) model, have been applied to this complicated system (e.g., Refs. [1–3]). Now, finding a minimum unit of complicated system is important for fundamental research and also for quantification of the system. For instance, particle physics studies the elementary constituents of matter under the high energy conditions; the number theory studies the distribution of prime numbers in a mathematical space. Then, is there a minimum unit of fracture patterns? Of course, we can say that all the geometrical patterns consist of dots. In this paper, we are not concerned with such a trivial solution, but the minimum unit that can extract the essence of fracture patterns and process including the phases of pre-nucleation, nucleation and propagation.

In order to find the geometrical minimum units of fracture patters, we use a discrete Walsh function and a lattice. Note that the concept of a lattice used in this paper is defined as a type of partially ordered set (see Section 3), and irrelevant to one used in the material science. The Walsh functions have been employed to measure symmetry and entropy of fracture patterns [4–6]. In physics, we are unfamiliar with the lattice, but it will be shown that this is closely related to the minimum patterns. As an application of our approach, we examine the temporal change of the spatial pattern of acoustic-emission (AE) events in a rock-fracturing experiment, published in Ref. [7]. In order to interpret what this examination means, we use

* Corresponding author. Tel.: +81 0788035737; fax: +81 788035757.
E-mail address: yk2000@kobe-u.ac.jp (K. Yamasaki).

0378-4371/$ – see front matter
doi:10.1016/j.physa.2008.07.014

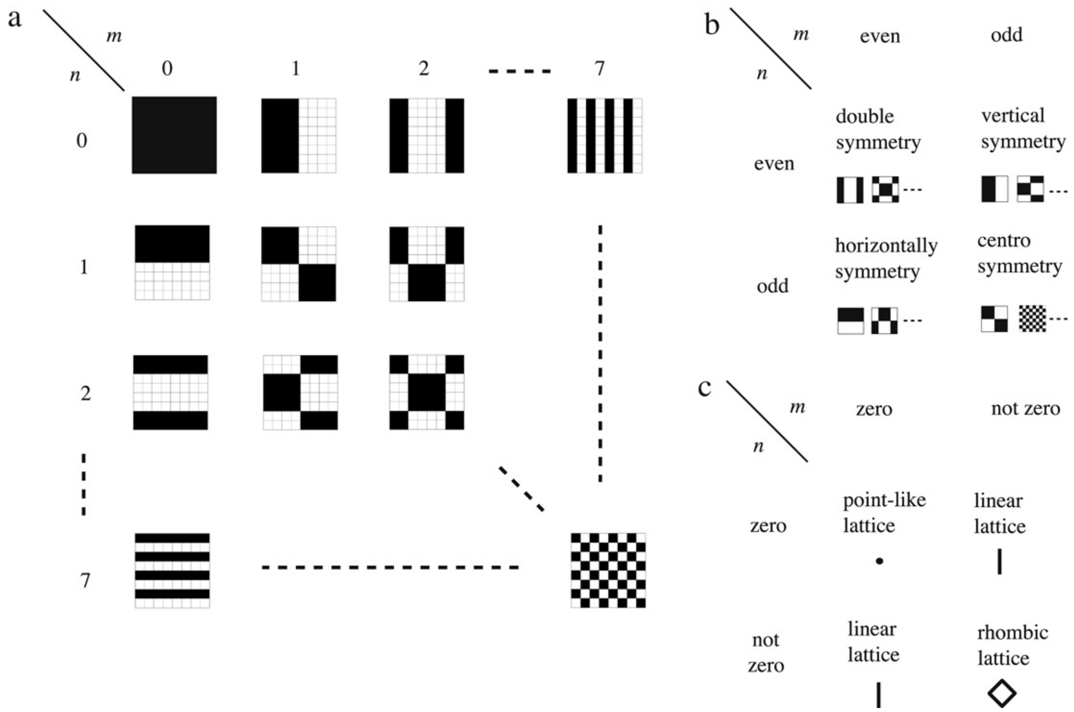

Fig. 1. (a) Examples of the two-dimensional discrete Walsh functions for $M = N = 8$. Black represents $+1$ and white represents -1. (b) Four types of symmetry defined by Eq. (3). Each symmetry has more than one expression, and the number of the expression increases with the increase of M and N. (c) Three types of symmetry (geometrical minimum units) defined by Eqs. (14)–(16). Each symmetry has only one expression, and the number of the expression keeps constant, 1.

a particular mathematical structure called sheaf on a lattice introduced by [8]. Our analysis is at present carried out for a two-dimensional (2D) space and it would be desirable to extend to 3D realizations.

The structure of this paper is as follows. In Section 2, we give a summary of the discrete Walsh functions applied to fracture patterns. In Section 3, we rewrite the Walsh functions in terms of the lattice, and suggest the minimum unit of fracture patterns. In Section 4, we introduce the symmetry and the entropy of the lattice, and apply them to the random and the fracture (AE) patterns. In Section 5, we discuss the minimum unit of fracture patterns and the application results from a viewpoint of the sheaf on a lattice.

2. Walsh functions and fracture patterns

In this paper, we use Walsh transforms for the quantification of fracture patterns [4–6]. Since Fourier transforms are more familiar than Walsh transforms, we use the Walsh transforms in terms of the Fourier (sine–cosine) functions, and give a summary of related mathematical techniques based on Ref. [9]. The order of procedure is as follows [9]. First, spatial pattern is considered as an information source consisting of four types of symmetry (Fig. 1(b)). Next, these symmetries emitted from the source are assumed to occur with the corresponding probabilities (Eqs. (5) and (6)). These probabilities can be estimated utilizing the discrete Walsh transform of the pattern. Finally, entropy function in information theory is applied to the probabilities so that we define entropy Eq. (7). Because this entropy is concerned with symmetry, it is called symmetropy.

Following Refs. [9–11], we represent Walsh function based on sinusoidal functions. The Walsh function $\mathrm{wal}(k, x)$ of order k and argument x can be represented over the interval $0 \le x < 1$ as follows: $\mathrm{wal}(k, x) = \prod_{i=0}^{m-1} \mathrm{sgn}[(\cos 2^i \pi x)^{ki}]$, where $k = 0, 1, \ldots$ and $k_i = 0$ or 1. The function $\mathrm{sgn}(t)$ is -1 if $t < 0$ and $+1$ if $t \ge 0$. The digits of the binary numeral for the integer k such that $k = \sum_{i=0}^{m-1} k_i 2^i$ and m is the smallest positive integer such that $2^m > k$. Fig. 2 shows the first five Walsh functions and the corresponding Fourier sinusoidal functions. Dyadic addition of nonnegative integers h and k is defined as $h \oplus k = \sum_{i=0}^{f} |h_i - k_i| 2^i$, where $h = \sum_{i=0}^{f} h_i 2^i$ and $k = \sum_{i=0}^{f} k_i 2^i$. Consequently, the product of the two Walsh functions is given by $\mathrm{wal}(h, x)\mathrm{wal}(k, x) = \mathrm{wal}(h \oplus k, x)$. The Walsh functions form a complete orthonormal set in the interval $0 \le x < 1$. Therefore, every function $f(x)$ which is integrable in the Lebesgue sense can be expressed in the interval $0 \le x < 1$ as a series of the form $f(x) = \sum_{i=0}^{\infty} a_i \mathrm{wal}(i, x)$, where the coefficients a_i are given by $a_i = \int_0^1 f(x)\mathrm{wal}(i, x)\mathrm{d}x$, for $i = 0, 1, \ldots$.

Next, we consider the discrete Walsh functions based on Ref. [9]. Let the interval (0, 1) be divided into $N = 2^q$ (q is a positive integer) with equal subintervals. The value of the nth order Walsh function in the ith subinterval is defined as