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Dynamical response to perturbation of critical Boolean networks

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ABSTRACT

Boolean networks can be used as simple but general models for complex self-organizing systems. The freedom to choose different rules and structures of interactions makes this model applicable to a wide variety of complex phenomena. It is known that the damage dynamics in annealed Boolean systems should fall in the same universality class of the directed percolation model. In this work we present results about the behavior of this model at and near the critically ordered condition for both the annealed and the quenched versions of the model. Our study concentrates on the way the system responds to a small perturbation. We show that the characteristic correlation time, i.e., the time in which any memory of this perturbation is lost, diverges as one moves towards criticality. Exactly at the critical point, we observe that the time for returning to the natural state after the perturbation follows a power-law distribution. This indicates that most perturbations are quickly restored, while few events may have a global effect on the system, suggesting a mechanism that assures at the same time robustness and adaptability. The critical exponents obtained are in agreement with the values expected for the universality class of mean-field directed percolation both in the annealed and in the quenched Boolean network model. This gives further evidence that annealed Boolean networks may in certain conditions provide a good model for understanding the behavior of regulatory systems. Our results may give insight into the way real self-organizing systems respond to external stimuli, and why critically ordered systems are often observed in Nature.

1. Introduction

Among the most relevant characteristics of complex systems is their property of spontaneously evolving towards highly organized states [1]. This means that the individual agents comprising the system have the ability to self-regulate their behavior to achieve a desired global condition. Moreover, ordering is achieved not by means of a precise structure of interaction between the agents, but as an emergent condition of the global dynamics. In other words, there are no central controllers or task distribution; each agent responds only to their local environment without complete knowledge of the system as a whole. Such a distributed design allows the system to efficiently correct local failures and respond to environmental changes.

Studies of models of complex systems have shown that by tuning a few parameters governing the interactions between the individual agents it is possible to make the system undergo a transition from an ordered to a chaotic state [2]. It has been speculated that neither the ordered nor the chaotic phase could display at the same time the robustness and adaptability observed in real regulatory systems, and these systems would more probably lie in the region of the phase space close to the critical condition [2]. It has also been proposed that the critical condition is often observed in Nature due to a dynamical

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control of the parameters that drives the system towards a critical self-organizing condition without the need of any fine-tuning [3]. Therefore, understanding the critical properties of these complex systems should be of fundamental importance in the study of their dynamics.

Boolean networks were introduced a few decades ago as a simple model for self-regulating systems [4]. Since then, this model has been applied in several areas including gene regulation [5,6], evolution [7], and neural networks [8] (see also Ref. [9] for a review). A Boolean network consists of a system of N nodes, whose states are described by a Boolean variable σ_i, that is, each node can assume only two possible states (0 or 1). The evolution takes place in discrete time steps, with all the nodes updating their states simultaneously in accordance with a node-specific Boolean function,

$$\sigma_i(t+1) = f_i[\sigma_{i1}(t), \sigma_{i2}(t), \ldots, \sigma_{ik_i}(t)], \tag{1}$$

where f_i is a Boolean function controlling the evolution of the ith node, and k_i is the number of neighbors that can influence the evolution of these nodes. In the random Boolean model, the two main parameters determining the dynamics of the system are the average number of connections $\langle K \rangle$ and bias ρ, which gives the probability with which a random input $\boldsymbol{\sigma}$ results in $f_i[\boldsymbol{\sigma}] = 1$. For simplicity, we consider here that every node has the same number of connections, that is $\langle K \rangle = K$. It has been demonstrated that in the random case the knowledge of these two parameters is enough to place the network in the ordered or chaotic regime [10,11].

In both ordered and chaotic cases the system evolves towards limiting cycles or attractors. However, in the ordered regime, after a perturbation the system typically returns to the same cycle, while in the chaotic regime there is finite probability that the system evolves to a different final cycle [12]. The behavior of this model near the critical regime has been the subject of some recent studies [13–15].

Here we study the behavior of random Boolean networks near the critical condition. We propose that the transition to the critical regime can be characterized by the divergence of the relaxation time t_R, that is, the time in which all memory of damage applied to the network is lost. With simple scaling arguments we show that, for the annealed system and at the critical condition, the cumulative probability distribution of t_R decays as a power law $P(t_r > t) \sim t^{-\beta}$, with an exponent $\beta = 1$. By means of extensive numerical simulations we confirm that this distribution follows the expected power law. Our results also show that finite network sizes result in an exponential truncation at time scales proportional to $N^{1/2}$. We also investigate the distribution of t_R when departing from the critical condition. At the ordered phase, we observe that above a characteristic time scale, the distribution presents a crossover to an exponential decay. In the same way, at the chaotic phase we observe in the distribution a crossover to a flat plateau; which is consistent with the divergence of the average relaxation time expected in the chaotic condition. In both cases we find that the onset of the crossover diverges when the system is set at the critical condition and the divergence is controlled by a characteristic exponent $\nu = 1$. Finally we investigate the effect of a quenched disorder in the critical exponents. We find that under certain conditions the same exponents can be observed in the quenched case.

2. Methods

One order parameter commonly used to characterize the phase in random Boolean networks is the average Hamming distance, that measures the number of nodes in different states in two replicas of the network. The two networks replicas should be identical in every regard but the state of the evolving nodes. The nodes that are in different states in the replicas correspond to the damaged fraction of the network. In the limit where the number of nodes goes to infinity, the Hamming distance should converge to zero at the ordered phase and to a positive value at the chaotic phase. It is conjectured that this thermodynamic limit can be modeled by the so called annealed approximation for Boolean networks [10]. In this approximation the network of contacts and the updating rules of the whole system are rebuilt at each time step. The damaging transitions of annealed systems are in the universality class of directed percolation [16]. In particular, for the case of random networks the damaging transition should fall into the universality class of the mean-field directed percolation model [17]. It was proposed [12] that the dynamics of Hamming distance should obey the following iterative map:

$$H(t+1) = I_1 \langle K \rangle H(t) + \frac{1}{2} I_2 \langle K(K-1) \rangle H(t)^2 + \cdots, \tag{2}$$

where the influence I_i is the probability that a unit with i damaged neighbors becomes damaged in the following time step. From Eq. (2) it is possible to approximate the rate of growth of the damage in the network.

$$\frac{\mathrm{d}H}{\mathrm{d}t} \approx (I_1 \langle K \rangle - 1) H(t) + \frac{1}{2} I_2 \langle K(K-1) \rangle H(t)^2 + \cdots. \tag{3}$$

For small damage, $H(t)$ should evolve as $H(t) = H_0 \mathrm{e}^{(I_1 \langle K \rangle - 1)t}$. The ordered phase is characterized by an exponential convergence of $H(t)$ to zero, while at the chaotic phase $H(t)$ diverges exponentially from zero. In between we have the critical condition that happens when $I_1 \langle k \rangle = 1$. In the particular case of random Boolean networks, the average influence does not depend on the number of damaged inputs, $I_1 = I_2 = I_j = I = 2\rho(1-\rho)$, thus reproducing the known critical condition [10]. At the edge of chaos, the first-order term on the right side of Eq. (3) vanishes, and instead of an exponential behavior, one finds a power-law decay,

$$H(t) \sim t^{-\beta}, \tag{4}$$