Physica A 387 (2008) 5073–5079

Contents lists available at ScienceDirect

Physica A

journal homepage: www.elsevier.com/locate/physa

Generalized relativistic Chapman–Enskog solution of the Boltzmann equation

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ARTICLE INFO

Article history:
Received 8 February 2008
Received in revised form 26 March 2008
Available online 13 May 2008

Keywords:
Relativistic kinetic theory
Chapman–Enskog method
Heat transfer

ABSTRACT

The Chapman–Enskog method of solution of the relativistic Boltzmann equation is generalized in order to admit a time-derivative term associated to a thermodynamic force in its first order solution. Both existence and uniqueness of such a solution are proved based on the standard theory of integral equations. The mathematical implications of the generalization introduced here are thoroughly discussed regarding the nature of heat as chaotic energy transfer in the context of relativity theory.

1. Introduction

The standard treatment of the Boltzmann equation (BE) provides microscopic support to the transport equations commonly used in science and engineering problems [1,2]. It is well known that whereas the local equilibrium solution to the homogeneous BE leads to the Euler equations of hydrodynamics, its first order correction in the Knudsen parameter yields the Navier–Stokes–Fourier transport equations of fluid mechanics for a dilute gas. In the standard non-relativistic method of establishing the first order correction to the local equilibrium function, all partial time derivatives are replaced with spatial gradients using the Euler equations. Both existence and uniqueness of the solutions built in this scheme have been shown using the theory of integral equations [3].

On the other hand, in relativity theory time is simply a new coordinate $ct = x^4$. According to this fact, the direct application of the conventional method used to obtain the first order correction to the local equilibrium function is shown here to lead to a first order correction to the distribution function in terms of both spatial and time derivatives of the local variables namely, the generalized four-component thermodynamic forces. This is done following the tenets of relativistic linear irreversible thermodynamics as shown in Ref. [4]. We wish to emphasize that in this work the Meixner version of non-equilibrium thermodynamics is developed. However, in most standard works on relativistic kinetic theory [5–8] these ideas are ignored. The consequence of the absence of time components in the structure of the thermodynamic forces is on the one hand, the obtention of transport equations which apparently violate causality, and on the other hand the tensorial structure of the heat flux in relativity, which is still a subject of debate [9]. *The main motivation of this work is precisely to critically analyze these fine points.*

The mathematical procedure used here to show that the solution obtained, considering four-component thermodynamic forces, exists and is unique is based on the standard theory of integral equations. One requires orthogonality between the

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doi:10.1016/j.physa.2008.05.012

inhomogeneous part of the equation and the collisional invariants. This relation implies the validity of Euler equations. Nevertheless the use of these equations is not required for the construction of the solution to first order in the gradients. These results have strong implications on the structure of the four-component heat flux as will be discussed in this work. Finally, the usual subsidiary conditions are invoked in order to adjust integration constants. This has, as will be discussed in this work, strong implications on the structure of the heat flux tensor.

The paper is divided as follows. In Section 2 the relativistic BE and some elements of the Chapman–Enskog method of solution are briefly reviewed. The general form of the first order in the gradients solution to the relativistic BE is proposed in Section 3 and both its existence and uniqueness are proved in Section 4. Finally, Section 5 includes a discussion of the implications of the results and some concluding remarks.

2. The Chapman–Enskog expansion

The starting point of the generalized formalism is the special relativistic Boltzmann equation for a simple system in the absence of external forces [5,6]:

$$v^{\alpha} f_{,\alpha} = J(f, f'), \tag{1}$$

which describes the evolution of the single particle distribution function $f = f(x^{\nu}, v^{\nu} | n(x^{\nu}, t), u(x^{\nu}, t), \varepsilon(x^{\nu}, t))$ namely, the molecular number density in phase space. In Eq. (1)

$$f_{,\alpha} = \begin{bmatrix} \dfrac{\partial f}{\partial x^k} \\ \dfrac{1}{c}\dfrac{\partial f}{\partial t} \end{bmatrix}, \tag{2}$$

and the index α, as well as all Greek indices in the rest of this work, runs from 1 to 4 while the Latin ones run up to 3. The local variables, number density n, hydrodynamic velocity u^k and internal energy ε per unit of mass are defined in terms of f as follows

$$n = \int f \gamma \, \mathrm{d}v^*, \tag{3}$$

$$n u^k = \int f \gamma v^k \mathrm{d}v^*, \tag{4}$$

$$n\varepsilon = mc \int f \gamma v^4 \mathrm{d}v^*. \tag{5}$$

Here v^{μ} is the molecular velocity four-vector which we denote in terms of the three velocity $\vec{w}$ as

$$v^{\alpha} = \begin{bmatrix} \gamma \vec{w} \\ \gamma c \end{bmatrix}, \tag{6}$$

where $\gamma = \left(1 - w^2/c^2\right)^{-1/2}$ is the usual relativistic factor. Also, the differential velocity element can be written as [7]

$$\mathrm{d}v^* = \gamma^5 \frac{c \mathrm{d}^3 w}{v^4} = 4\pi c^3 \sqrt{\gamma^2 - 1} \mathrm{d}\gamma, \tag{7}$$

and, as usual, c is the speed of light. Eq. (1) implies that changes in the distribution function are due to collisions, represented on the right side of Boltzmann's equation through the so-called collision kernel $J(f, f')$ which is given by

$$J(f, f') = \iint \left(f' f_1' - f f_1\right) \sigma(\Omega) g \mathrm{d}\Omega \mathrm{d}v_1^*. \tag{8}$$

In Eq. (8) primes denote quantities after a binary collision between particles with velocity v and v_1 takes place; $\sigma(\Omega)\mathrm{d}\Omega$ is the differential cross section element and g is the relative velocity.

To solve Eq. (1) following what is now known in the literature as the Chapman–Enskog method [1], originally due to Hilbert, the distribution function is expanded in a power series of the Knudsen parameter λ around the local equilibrium distribution function $f^{(0)}$

$$f = f^{(0)} \left(1 + \lambda \phi^{(1)} + \lambda^2 \phi^{(2)} + \cdots\right). \tag{9}$$

The expansion parameter λ is a measure of the relative magnitude of the gradients of the local variables within a mean free path and the characteristic size of the system. The local equilibrium distribution function, solution to the homogeneous