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Physica A 387 (2008) 4411–4416

PHYSICA A

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Topology and dynamics of attractor neural networks: The role of loopiness

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Received 23 October 2007; received in revised form 26 January 2008
Available online 4 March 2008

Abstract

We derive an exact representation of the topological effect on the dynamics of sequence processing neural networks within signal-to-noise analysis. A new network structure parameter, loopiness coefficient, is introduced to quantitatively study the loop effect on network dynamics. A large loopiness coefficient means a high probability of finding loops in the networks. We develop recursive equations for the overlap parameters of neural networks in terms of their loopiness. It was found that a large loopiness increases the correlation among the network states at different times and eventually reduces the performance of neural networks. The theory is applied to several network topological structures, including fully-connected, densely-connected random, densely-connected regular and densely-connected small-world, where encouraging results are obtained.

PACS: 87.10.+e; 89.75.Fb; 87.18.Sn; 02.50.-r

Keywords: Complex networks; Neural networks; Loopiness; Probability theory

Starting with the pioneering work of modelling complex networks [1,2], the research area related to complex networks has been growing very fast. In parallel with studies of their structural properties, there has been a growing interest in dynamic systems defined on networks, for example, synchronization and collective dynamics, epidemic spreading, cascading failures, opinion formation, also various strategic games and some physical models such as the Ising model (see Ref. [3] and Refs. therein).

Neural assemblies (i.e. local networks of neurons transiently linked by selective interactions) are considered to be largely distributed and linked to form a web-like structure in the brain. Many researchers suggest that neural connectivity is far more complex than the random graph. The cortical neural networks of Caenorhabditis elegans and cat were reported to be small-world (SW) and scale-free (SF), respectively [1,4]. It is very important to understand how the complex neural wiring architecture is related to brain functions With the same average connections, a Hopfield network with random topology was reported to be more efficient for storage and recognition of patterns than either an SW network or a regular network [5,6]. For SF connections, with the same number of synapses, Torres et al. found that the capacity is larger than the storage of the highly diluted random Hopfield networks [7]. Using a Monte-Carlo

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doi:10.1016/j.physa.2008.02.073

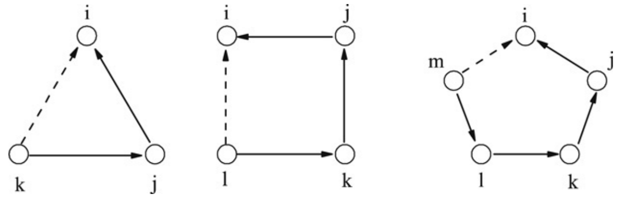

Fig. 1. Left: the first-order loopiness coefficient, L_1, means the probability of connectivity between k and i when j is connected to i and k is connected to j. Middle: the second-order loopiness coefficient L_2. Right: the third-order loopiness coefficient L_3.

method to lower the clustering coefficient smoothly with the degree of each neuron kept unchanged, Kim found that the networks with lower clustering exhibit much better performance [8].

It is well known that the equilibrium properties of fully-connected Hopfield networks have been extensively studied using spin-glass theory, especially the replica method [9–11]. The dynamics of fully-connected Hopfield models with static patterns and sequence patterns were widely studied using generating functional analysis [12,13]. As a simple relaxation of the biological unrealistic fully-connected model, various randomly diluted models were studied, including an extreme diluted model [14,15], a finite diluted model [16,17] and a finite connection model [18]. However, in the case of complex network topology, as far as we know, there are hardly any theoretical studies for dynamics or statics.

In this paper we use signal-to-noise analysis to study the effect of topology on the transient dynamics of sequence processing neural networks. For mathematical convenience we only focus on sequence processing models here and the relationship between our results and Hopfield models will be discussed.

The topological effects on neural networks mainly come from loops of topology [19]. In the case of so-called extremely diluted structures ($\lim_{N\to\infty} \bar{k}^{-1} = \lim_{N\to\infty} \bar{k}/N = 0$), the average loop length is very big, i.e. $\log_{k-1} N$, so the number of short loops, such as triangles or quadrangles, in the networks will be very small and the effect caused by short loops can be neglected. The dynamics of networks in this case is easy to study because at different time steps each spin is uncorrelated [14]. By contrast, if such small loops do exist, the correlations and feedbacks in a network will lead to more complicated dynamics.

In order to quantitatively present the effect of loops we define a new parameter, the loopiness coefficient, to represent the probability of finding loops in the network. The definition is shown in Fig. 1. In the left triangle spin j is connected to spin i and spin k is connected to spin j. We define the probability that k is connected to i as the first-order loopiness coefficient, L_1. Similarly, the nth-order loopiness coefficient, L_n, denotes the linking probability between two vertices to form a loop with $n+2$ edges.

We now study a sequence processing model consisting of $N \to \infty$ Ising-type neurons $s_i(t) \in \{+1, -1\}$. The neurons update their states simultaneously, with the following probabilities,

$$\mathrm{Prob}[s_i(t+1)\,|h_i(t)] = \frac{e^{\beta s_i(t+1) h_i(t)}}{2\cosh(\beta h_i(t))}, \tag{1}$$

where the local field $h_i(t) = \sum_{j=1}^{N} J_{ij} s_j(t)$, and β is the inverse temperature. For the transfer function $g(\cdot)$, we denote by $s_i(t+1) = g(h_i(t))$.

Let us store $p = \alpha N$ random patterns $\xi^\mu = (\xi_1^\mu, \ldots, \xi_N^\mu)$ in the network, where α is the loading ratio. So the interaction matrix $J_{ij} = \frac{c_{ij}}{Nc}\sum_{\mu=1}^{p} \xi_i^{\mu+1}\xi_j^\mu$ is chosen to retrieve the patterns as $\xi^1 \to \xi^2 \to \cdots \to \xi^p \to \xi^1$ (note that $\xi^{p+1} = \xi^1$), where c_{ij} is the adjacency matrix ($c_{ij} = 1$ if j is connected to i, $c_{ij} = 0$ otherwise) [20]. Consequently, the degree of spin i is $k_i = \sum_{j\in T_i} c_{ij} = \sum_j c_{ij}$, where T_i is the set of spin j connected with i. In this work, for studying the role of loopiness, we only consider $k_i \approx \bar{k} = Nc$. We assume that this property holds for dense connected random networks, dense connected SW networks, and dense connected regular networks.

For any pattern ξ^ν, the order parameter is $m^\nu(t) = \frac{1}{N}\sum \xi_i^\nu s_i(t)$ which represents the overlap between $\mathbf{s}(t)$ and the condensed pattern ξ^ν. The local field in neuron i is described by

$$h_i(t) = \frac{1}{Nc}\sum_{j\in T_i} \xi_i^{\nu+1}\xi_j^\nu s_j(t) + Z_i(t), \tag{2}$$