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Synchronization analysis for nonlinearly-coupled complex networks with an asymmetrical coupling matrix

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Abstract

In this paper, the global synchronization for an array of nonlinearly coupled identical chaotic systems is investigated. A distinctive feature of this work is to address synchronization issues for nonlinearly coupled complex networks with an asymmetrical coupling matrix. By projecting the nonlinear coupling function onto a linear one and assuming the difference between them as a disturbing function, we give some criteria for the global synchronization in virtual of the left eigenvector corresponding to the zero eigenvalue of the coupling matrix. Numerical examples are also provided to demonstrate the effectiveness of the theory.

Keywords: Global synchronization; Asymmetrical coupling matrix; Nonlinear coupling function; Projection method; The left eigenvector

1. Introduction

The topic of synchronization for coupled oscillators [1] has attracted numerous scientists from diverse fields including physics, biology, neuroscience, mathematics, chemistry and ecology, etc. Synchronization can be understood as the adjustment of rhythms or coherence of states by interaction (see [2]). Historically, synchronization phenomena have been actively investigated since the earlier days of physics. In the 17th century, Huygens discovered that two pendulum clocks hanging at the same beams were able to synchronize their phase oscillations, and other examples of synchronization include the synchronized lightning of fireflies [3], and the properties of adjacent organ pipes which can almost reduce one another to silence or speak in absolute unison.

Hitherto, many different synchronization phenomena and models have been studied; namely, complete or identical synchronization, phase synchronization, cluster synchronization, and so on. For example, in [4], the author uses the properties of invariant manifold (normal k-hyperbolicity) to describe synchronization; in [5], the authors investigate the destabilization (oscillatory behaviour) problem in a network of identical globally asymptotically stable systems in case the isolated systems are nonminimum phase, and in [6,7], the boundedness and synchronization problems for linearly coupled oscillators are considered via the semi-passivity property; in [8], the authors present a master stability function based on the transverse Lyapunov exponents to study local synchronization; in [9,10], a distance from synchronization manifold to each state is defined to study the global synchronization; in [11–14], the left

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eigenvector corresponding to the zero eigenvalue of the diffusive coupling matrix is utilized to investigate the global synchronization; and the coupling matrix can also be regraded as the topological structure of the network, so in [15–17], using graph theory, connection graph stability method is established. Moreover, in [18–21] and references therein, synchronization of randomly connected complex networks such as scale-free and small-world networks and other forms like stochastic coupled networks are studied.

However, previous studies of synchronization mainly focus on oscillators under linear coupling, with the coupling matrix (or connection topology) constant, time varying or state dependent, see [9–13,15–17], etc. Even in those papers investigating nonlinear protocols, like [14,22,23], there exists a strong assumption on the complex networks: the interaction topology should be bidirectional, i.e. the coupling matrix should be symmetrical. However, unidirectional communication is important in practical applications and can be easily incorporated, for example, via broadcasting. Also, sensed information flow which plays a central role in schooling and flocking is typically not bidirectional.

In this paper, we will investigate the synchronization of nonlinearly coupled complex networks, described by ordinary differential equations (ODE). One feature sharply distinguishing from model reported in literature is that in this paper the coupling matrix is asymmetrical and the coupling functions are nonlinear. Our approach is to project the nonlinear coupling function onto a linear one and assume the difference between them as a disturbing function. By using the left eigenvector corresponding to the zero eigenvalue of the coupling matrix, we derive some criteria to ensure the global synchronization.

An outline of this paper is as follows. In Section 2, a model of nonlinearly-coupled complex networks is proposed. In Section 3, some necessary definitions, lemmas and hypotheses are given. In Section 4, based on the projection method, some criteria for the global synchronization of nonlinearly-coupled systems are derived. Some discussions and comparisons with previous works are given in Section 4. In Section 5, numerical examples are presented to show the validity of the theoretical analysis. Finally, we conclude this paper in Section 6.

Notations: Throughout this paper, we denote the vector $(1, \dots, 1)^{\mathrm{T}}$ by $\mathbf{1}$. The identity matrix is denoted by I. The dimension of these vectors and matrices will be cleared in the context. If all eigenvalues of a matrix $A \in R^{n\times n}$ are real, then we sort them as $\lambda_1(A) \le \lambda_2(A) \le \cdots \le \lambda_n(A)$. And we denote the symmetrical part of A as $A^s = (A + A^{\mathrm{T}})/2$. A symmetric real matrix A is positive definite (semi-definite) if $x^{\mathrm{T}}Ax > 0(\ge 0)$ for all nonzero x. We denote this as $A > 0(A \ge 0)$. The Kronecker product of an n by m matrix $A = (a_{ij})$ and a p by q matrix B is the np by mq matrix $A \otimes B$, defined as

$$A \otimes B = \begin{pmatrix} a_{11}B & \cdots & a_{1m}B \\ \vdots & \ddots & \\ a_{n1}B & & a_{nm}B \end{pmatrix}$$

and the Kronecker product has such property: $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$.

2. Model description

In general, we denote the dynamics for each isolated (uncoupled) node as

$$\dot{x}^i(t) = f(x^i(t)) \tag{1}$$

where $x^i = (x^i_1, \dots, x^i_n)^{\mathrm{T}} \in R^n$ denotes the state variable vector of the i-th node, the intrinsic function $f(\cdot) = (f_1(\cdot), \dots, f_n(\cdot))^{\mathrm{T}} : R^n \to R^n$ is continuous. N nonlinearly-coupled systems can be described as

$$\dot{x}^i(t) = f(x^i(t)) - \sigma \sum_{j=1}^{N} l_{ij} h(x^j(t)); \quad i = 1, \dots, N \tag{2}$$

where $\sigma > 0$ is the coupling strength; the diffusive coupling matrix $L = (l_{ij})$ satisfies: $l_{ij} \le 0, i \ne j$ and $l_{ii} = -\sum_{j=1, j\ne i}^{N} l_{ij}$, which can be also regarded as the graph Laplacian if we assume the coupled complex networks as a weighted directed graph; and the nonlinear coupling function $h(\cdot) : R^n \to R^n$ is continuous and has the form: $h(x^i(t)) = (h_1(x^i_1(t)), \dots, h_n(x^i_n(t)))^{\mathrm{T}}, i = 1, \dots, N$.

In this paper, we will investigate the complete synchronization, defined as

$$\lim_{t\to\infty} \|x^i(t) - x^j(t)\| = 0 \quad i, j = 1, \dots, N. \tag{3}$$