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Asymptotic behavior of the supremum tail probability for anomalous diffusions

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Abstract

In this paper we investigate asymptotic behavior of the tail probability for subordinated self-similar processes with regularly varying tail probability. We show that the tail probability of the one-dimensional distributions and the supremum tail probability are regularly varying with the pre-factor depending on the moments of the subordinating process. We can apply our result to the so-called anomalous diffusion.

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1. Introduction

Continuous time random walk (CTRW) was introduced in pioneering works by Sher, Montroll and Weiss [1,2]. Since then CTRW has found widespread applications in many scientific fields such as physics, finance and insurance. There are many models which replace CTRW by a certain diffusion. In this paper we investigate processes which can serve as an approximation of CTRW. We will treat the so-called subordinated process X, that is,

$$X(t) = Z(V(t)) \tag{1}$$

where Z and V are stochastic processes defined on the same probability space and the process Z is called a main process and the process V is a subordinating process which should take non-negative values and its trajectories are non-decreasing. Subordinated processes are employed in the description of anomalous dynamics, [3–6]. If Z is an α-stable Lévy process the process X is the so-called anomalous diffusion, [7–10]. For a comprehensive study of stable distributions and processes see Refs. [11,12]. The aim of this paper is to derive the asymptotic behavior of the tail probability of the one-dimensional distributions and the tail probability of supremum over finite interval for the process X. As an application of our result we can consider the anomalous diffusions which appear, for example, in modeling risk process, [13] or in the Cole–Cole relaxation responses, [14]. As a special case we get also the

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asymptotic behavior of the so-called finite time ruin probability. The finite time ruin probability for an anomalous diffusion is treated in Ref. [13]. The one-dimensional probability density of anomalous diffusions is investigated in Ref. [10]. The asymptotic tail behavior of the one-dimensional distributions of the anomalous diffusion when the main process is a symmetric α-stable process is given in Ref. [10].

2. Main result

We will write $g(u) \cong h(u)$ if $\lim_{u\to\infty} \frac{g(u)}{h(u)} = 1$. Let us recall that a real-valued stochastic process Z is self-similar with index $H > 0$ if $\{Z(at), t \in [0,\infty)\} \stackrel{d}{=} \{a^H Z(t), t \in [0,\infty)\}$ for all $a > 0$ where $\stackrel{d}{=}$ denotes equality of the finite-dimensional distributions.

Theorem 1. *Let Z be a self-similar process with the parameter of self-similarity $H > 0$ and*

$$\mathbb{P}(\sup_{t\le 1} Z(t) > u) \cong \mathbb{P}(Z(1) > u) \tag{2}$$

where the distribution of $Z(1)$ is regularly varying with index $\theta > 0$, that is,

$$\lim_{u\to\infty} \frac{\mathbb{P}(Z(1) > wu)}{\mathbb{P}(Z(1) > u)} = w^{-\theta} \tag{3}$$

for all $w \ge 1$. Moreover, let V be a stochastic process with non-negative values, non-decreasing sample paths, defined on the same probability space as the process Z, independent of Z and $0 < \mathbb{E}V^{H\theta+\delta}(1) < \infty$ for some $\delta > 0$. Then

$$\mathbb{P}(\sup_{t\le 1} Z(V(t)) > u) \cong \mathbb{P}(Z(V(1)) > u) \cong \mathbb{E}V^{H\theta}(1)\,\mathbb{P}(Z(1) > u). \tag{4}$$

Proof. First let us consider a lower bound. Note that

$$\begin{aligned}\mathbb{P}(\sup_{t\le 1} Z(V(t)) > u) &\ge \mathbb{P}(Z(V(1)) > u)\\ &= \mathbb{P}(V^H(1)\,Z(1) > u)\\ &\cong \mathbb{E}V^{H\theta}(1)\mathbb{P}(Z(1) > u)\end{aligned}$$

where in the second line we used the self-similarity property of the process Z and in the last line we can apply Corollary 3.6 of Ref. [15] (see also [16]) because $\mathbb{E}V^{H\theta+\delta}(1) < \infty$ for some $\delta > 0$ and $\mathbb{P}(Z(1) > u)$ is regularly varying with index θ. Corollary 3.6 of Ref. [15] is on asymptotic behavior of the tail distribution for the product of independent random variables.

Now we investigate the upper bound

$$\begin{aligned}\mathbb{P}(\sup_{t\le 1} Z(V(t)) > u) &\le \mathbb{P}(\sup_{t\le V(1)} Z(t) > u)\\ &= \int_0^\infty \mathbb{P}(\sup_{t\le s} Z(t) > u)\,\mathrm{d}F_{V(1)}(s)\\ &= \int_0^\infty \mathbb{P}(\sup_{t\le 1} Z(st) > u)\,\mathrm{d}F_{V(1)}(s)\\ &= \int_0^\infty \mathbb{P}(\sup_{t\le 1} s^H Z(t) > u)\,\mathrm{d}F_{V(1)}(s)\\ &= \mathbb{P}(V^H(1)\sup_{t\le 1} Z(t) > u)\\ &\cong \mathbb{E}V^{H\theta}(1)\,\mathbb{P}(\sup_{t\le 1} Z(t) > u)\\ &\cong \mathbb{E}V^{H\theta}(1)\,\mathbb{P}(Z(1) > u)\end{aligned}$$

where in the second last line we use Corollary 3.6 of Ref. [15] (see also [16]) and (2) and (3) and the last line follows from (2) and (3). □