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Physica A 386 (2007) 720–728
PHYSICA A
www.elsevier.com/locate/physa

Roundoff-induced attractors and reversibility in conservative two-dimensional maps

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Available online 9 August 2007

Abstract

We numerically study two conservative two-dimensional maps, namely the baker map (whose Lyapunov exponent is known to be positive), and a typical one (exhibiting a vanishing Lyapunov exponent) chosen from the generalized shift family of maps introduced by C. Moore [Phys. Rev. Lett. 64 (1990) 2354] in the context of undecidability. We calculate the time evolution of the entropy $S_q \equiv (1-\sum_{i=1}^{W} p_i^q)/(q-1)$ ($S_1 = S_{BG} \equiv -\sum_{i=1}^{W} p_i \ln p_i$). We exhibit the dramatic effect introduced by numerical precision. Indeed, in spite of being area-preserving maps, they present, *well after* the initially concentrated ensemble has spread virtually all over the phase space, unexpected *pseudo-attractors* (fixed-point like for the baker map, and more complex structures for the Moore map). These pseudo-attractors, and the apparent time (partial) reversibility they provoke, gradually disappear for increasingly large precision. In the case of the Moore map, they are related to zero Lebesgue-measure effects associated with the frontiers existing in the definition of the map. In addition to the above, and consistent with the results by V. Latora and M. Baranger [Phys. Rev. Lett. 82 (1999) 520], we find that the rate of the far-from-equilibrium entropy production of baker map numerically coincides with the standard Kolmogorov–Sinai entropy of this strongly chaotic system.

Keywords: Nonlinear dynamics; Nonextensive statistical mechanics; Precision effects; Attractors; Weak chaos

1. Introduction

Dynamical systems which present exponential sensitivity to the initial conditions are called chaotic (*also called strongly chaotic*). Indeed, almost all orbits are unpredictable for any finite-precision calculation, even if the evolution was purely deterministic. But dynamical systems at the edge of chaos are also somehow unpredictable, they typically exhibit a power-law sensitivity to the initial conditions, and are called *weakly chaotic*. For both cases, the sensitivity ξ to the initial conditions is typically given [1–3] by

$$\xi(t) \equiv \lim_{\Delta \mathbf{x}(0)\to 0} \frac{|\Delta \mathbf{x}(t)|}{|\Delta \mathbf{x}(0)|} = [1+(1-q)\lambda_q t]^{1/(1-q)}, \tag{1}$$

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0378-4371/$ - see front matter
doi:10.1016/j.physa.2007.07.070

where $\Delta\mathbf{x}(t)$ is the time-dependent discrepancy between two initially close trajectories. The coefficient λ_q is a generalized Lyapunov exponent, and q is an index associated with the entropy [4]

$$S_q \equiv \frac{1-\sum_{i=1}^{W} p_i^q}{q-1} \quad \left(q \in \mathcal{R}; S_1 = S_{BG} \equiv -\sum_{i=1}^{W} p_i \ln p_i\right), \tag{2}$$

where *BG* stands for *Boltzmann–Gibbs*. This entropy is at the basis of *nonextensive statistical mechanics*, which has received many applications for complex systems. Eq. (1) recovers, in the $q=1$ limit, the usual exponential divergence $\xi = e^{\lambda_1 t}$ (λ_1 being, say for the simple one-dimensional case, the standard Lyapunov exponent). If $q=0$, we obtain the simple result $\xi \propto t$. In general, $q \neq 1$ yields a power-law dependence.

A meaningful statistical description is possible even when the maximal Lyapunov exponent vanishes. This fact has been illustrated in the Casati–Prosen triangle map [5], a mixing and ergodic conservative system which presents the extreme case of linear instability [5]. This map satisfies that, in the infinitely fine graining limit (i.e., $W \to \infty$), the q-entropy increases linearly with time only for the value of the entropic index $q=0$ [6]; its slope is expected to coincide with the q-generalized Kolmogorov–Sinai entropy rate κ_q [1]. Furthermore, this value is expected to coincide with the q-generalized Lyapunov coefficient; in other terms, a Pesin-like equality [7] is expected to hold for generic q as well.

Conservative dynamical systems leading to entropic indices $q \neq 1$ are certainly interesting. The case that we have just mentioned, namely the Casati–Prosen map (for which $q=0$), is one such example. In the present paper we study various aspects along this direction, possibly with q different from both unity and zero.

We are interested in studying the *entropy production*, i.e., the rate of increase of the q-entropy, of two-dimensional maps and, if possible, to connect it with the Kolmogorov–Sinai entropy rate [8] κ_1, which is a property defined for (strongly) chaotic dynamical systems. In this paper we present numerical results for the non-dissipative baker map, and for one (from now on referred to as *Moore map*) of the shift-like dynamical systems proposed by Moore [9]. The latter is expected to present a power-law sensitivity to the initial conditions, i.e., $q<1$ (possibly $q \neq 0$, in contrast with the Casati–Prosen map).

To evaluate the q-entropy of the system, we first partition the phase space into $W \gg 1$ little equal cells; we then choose one of these cells and put within $N \gg 1$ random initial conditions. As time t evolves, the N points spread over the phase space in such a way that, at each time t, we have a set of numbers $\{N_i(t)\}$ ($\sum_{i=1}^{W} N_i(t) = N$, $\forall t$), so that $N_i(t)$ is the number of points inside the ith cell. Then, for each value of t, we can consider a set of probabilities $\{p_i(t) \equiv N_i(t)/N\}$ to find N_i points in the ith cell. For achieving a numerically meaningful definition of the probabilities p_i, the condition $N \gg W$ has to be fulfilled; we typically consider $N = 10W$. At $t=0$, all probabilities but one are zero, hence $S_q(0)$, calculated through Eq. (2), vanishes $\forall q$. In other words, before the system starts to evolve, we know all the information regarding the occupancy of the phase space. As t evolves, information is lost and $S_q(t)$ starts to increase. In all cases, S_q will be bounded by its corresponding equiprobability value $(S_q)_{max} = (W^{1-q}-1)/(1-q)$, whose $q=1$ limit case yields $\ln W$. The q-entropy production is then defined as

$$K_q \equiv \lim_{t\to\infty} \lim_{W\to\infty} \lim_{N\to\infty} \frac{S_q(t)}{t}. \tag{3}$$

In practice, we take values of (N, W, t) large enough so that our numerical result for K_q becomes independent from them.

Summarizing, we numerically study direct snapshots of the occupancy of the space phase, and the time evolution of $S_q(t)$. From these, we shall present that:

(i) $K_1 = \kappa_1$ for the baker map;
(ii) numerical precision plays a relevant role in the phase space occupancy and time evolution of $S_q(t)$ in both baker and Moore maps;
(iii) the time evolution of S_q partially (but not completely) reflects the time evolution of the phase space occupancy in both baker and Moore maps;
(iv) $q \neq 1$ for the Moore map.