Physica A 388 (2009) 4887–4892

Contents lists available at ScienceDirect

Physica A

journal homepage: www.elsevier.com/locate/physa

On the construction of complex networks with optimal Tsallis entropy

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ARTICLE INFO

Article history:
Received 24 April 2009
Received in revised form 17 July 2009
Available online 12 August 2009

PACS:
87.14.Gg
87.15.Aa
87.23.Kg
89.75.K

Keywords:
Tsallis entropy
Growing networks
Scale-free networks

ABSTRACT

In this work, we first formulate the Tsallis entropy in the context of complex networks. We then propose a network construction whose topology maximizes the Tsallis entropy. The growing network model has two main ingredients: copy process and random attachment mechanism (C-R model). We show that the resulting degree distribution exactly agrees with the required degree distribution that maximizes the Tsallis entropy. We also provide another example of network model using a combination of preferential and random attachment mechanisms (P-R model) and compare it with the distribution of the Tsallis entropy. In this case, we show that by adequately identifying the exponent factor q, the degree distribution can also be written in the q-exponential form. Taken together, our findings suggest that both mechanisms, copy process and preferential attachment, play a key role for the realization of networks with maximum Tsallis entropy. Finally, we discuss the interpretation of q parameter of the Tsallis entropy in the context of complex networks.

1. Introduction

From the discovery that many real networks exhibit universal properties and because of the intriguing nature of these discrete large-scale complex systems, network theory has gained the enthusiasm of many physicists. In recent years, network analyses and modelling have shown an increasing prominence and popularity within multi-disciplinary fields from social and biological sciences to applied engineering in communication and transportation systems [1–5].

Many real networks can be classified into random or scale-free networks by using a statistical measure called degree distribution $P(k)$. The node degree distribution $P(k)$ is the probability for finding a node with degree k in the network. In the case of random networks, the degree distribution is a Poissonian [6]. However, most of the real networks deviate from the random graph statistics and seem to follow a power-law distribution $k^{-\gamma}$, where γ is the degree exponent for each network. These networks are called scale-free networks [7]. This sort of universality has encouraged many physicists to seek an understanding of the emergence of complex networks based on first and fundamental principles.

On the other hand, physicists have been fascinated during decades by the prediction power and applicability of Thermodynamics. This branch of Physics is essentially based on the energy and entropy concepts. While the energy is usually related to the physical system and can be associated to a Hamiltonian, the interpretation of the entropy is a bit more complicated but it is usually associated to the information upon the physical system [8]. The standard thermodynamics and the Boltzmann–Gibbs (BG) statistical mechanics have been used for decades to characterize and explain a large variety of physical systems. Furthermore, it was believed that the microscopic expression of the entropy had to be unique (i.e., system independent). However, over the past two decades it has been reported that several systems, involving long-range interactions and dissipative processes in particular, showed anomalies when BG statistics were computed [8–10].

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0378-4371/$ – see front matter
doi:10.1016/j.physa.2009.07.049

Then, nonextensive statistical mechanics was introduced as a way to generalize the BG theory and was successful in addressing a wide range of nonequilibrium phenomena in complex systems [11]. The entropy associated to the nonextensive statistical mechanics was called Tsallis entropy [9].

Although many works on network analyses have been done in recent years, not many of them have focused on the connection between nonextensive statistical physics and network theory [12–14]. Moreover, there are currently several well-established models and mechanisms that are able to generate scale-free topologies [1–5,15]. However, network models that simultaneously generate power-law degree distribution and that are able to maximize the Tsallis entropy have not been sufficiently investigated.

In this work, we first formulate the Tsallis entropy in the context of complex networks and define the energy of the system as a function of the node degree. We then propose a model that constructs a network whose topology maximizes the Tsallis entropy. The growing network model contains two main ingredients: copy process and random attachment mechanism (C-R model). We show that the resulting degree distribution of the C-R model exactly agrees with the required degree distribution that maximizes the Tsallis entropy. We also give another example of network model, generated using a combination of Barabasi–Albert model [7] with preferential and random attachment processes (P-R model), and compare it with the distribution of the Tsallis entropy. In this case, we show that by adequately identifying the exponent factor q, the degree distribution can also be written in the q-exponential form. Our findings suggest that both the copy process and preferential attachment mechanism play a key role for the realization of networks with maximum Tsallis entropy [9]. Finally, we discuss the interpretation of q parameter of the Tsallis entropy in the context of complex networks.

2. Tsallis entropy

In this section, we first review the concept of Tsallis entropy defined as an extension of Shannon entropy in the context of networks [9]. Then, we will derive the optimal degree distribution which maximizes the Tsallis entropy, corresponding to a complex network whose q-average node degree is constant.

Let us consider the Tsallis entropy formulation in the context of complex networks. Let p_k be the probability for finding a node with degree k. Then, the degree distribution probability is normalized as follows:

$$\sum_k p_k = 1, \tag{1}$$

which plays a role as a constraint of Tsallis entropy. Although in the Tsallis entropy formalism there are several ways of constraining the energy, we use the following constraint (See Ref. [10]):

$$\sum_k \epsilon_k p_k^q = E. \tag{2}$$

Later, we will see that this implies that the q-average node degree is constant.

In the Tsallis entropy formalism, the generalized form of the entropy can be written as follows (see Ref. [10]):

$$S_q = -\frac{\sum_k p_k^q - 1}{q-1}. \tag{3}$$

It is known that q-exponential is defined as follows (For example, see Ref. [10]):

$$e_q^x = \exp_q(x) = (1 + (1-q)x)^{1/(1-q)}. \tag{4}$$

Maximizing the entropy (3) under the constraints (1) and (2), the following expression is obtained:

$$p_k \propto (1 + \beta'(q-1)\epsilon_k)^{-\frac{1}{q-1}} = \exp_q(-\beta'\epsilon_k), \tag{5}$$

where β' is a Lagrangian multiplier.[1] This is well known as the Tsallis entropy distribution which exhibits q-exponential form.

Let us define the energy ϵ_k of each node with degree k as follows:

$$\beta'\epsilon_k = k - 1. \tag{6}$$

Considering (2), this implies that a network has a constant q-average node degree.

Inserting (6) into (5), the following expression is obtained:

$$p_k \propto (1 + (q-1)(k-1))^{-\frac{1}{q-1}} = \exp_q(-(k-1)). \tag{7}$$

[1] We use the notation β' for a Lagrangian multiplier, because we will use β in a different context. In particular, we will use β for the ratio of random attachment in C-R model and P-R model.