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Physica A 371 (2006) 96–99
PHYSICA A
www.elsevier.com/locate/physa

From coupled map lattices to the stochastic Kardar–Parisi–Zhang equation

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Available online 19 May 2006

Abstract

We discuss the space and time dependence of the continuum limit of an ensemble of coupled logistic maps on a one-dimensional lattice. We show that the resulting partial differential equation has elements of the stochastic Kardar–Parisi–Zhang growth equation and of the Fisher–Kolmogorov–Petrovskii–Piscounov equation describing front propagation. A similar study of the Lyapunov vector confirms that its space–time behaviour is of KPZ type.

Keywords: Coupled logistic maps; Stochastic growth; Travelling waves; Lyapunov vector

1. Introduction

Coupled map lattices are dynamical systems with very different collective spatio-temporal regimes selected by tunning a few parameters that are, typically, a local parameter controlling the chaoticity of the independent units, and the coupling between different units [1]. There have been several attemps to use statistical mechanics notions to describe their spatio-temporal behaviour [2]. Continuous limits have also been considered [3]. In this contribution, we summarize the results of our recent investigation of the continuous limit of a one-dimensional ring of diffusively coupled logistic maps [4]. We discuss its connection with (i) the stochastic Kardar–Parisi–Zhang (KPZ) growth partial-differential equation [5]; (ii) the deterministic Fisher–Kolmogorov–Petrovskii–Piscounov (FKPP) partial differential equation for the spreading of a favourable mutation in the form of a wave [6,7].

2. The continuum limit of the coupled map lattice

The logistic map is a non-linear evolution equation acting on a continuous variable x taking values in the unit interval $[0, 1]$:

$$x_n = f(x_{n-1}) \equiv r x_{n-1}(1 - x_{n-1}). \tag{1}$$

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0378-4371/$ - see front matter
doi:10.1016/j.physa.2006.04.083

The discrete time is labelled by $n = 0, 1, \ldots$. The parameter r takes values in $[0, 4]$. The time series has very different behaviour depending on the value of r. For $0 \leqslant r < 1$ the iteration approaches the fixed point $x^* = 0$. For $1 \leqslant r < 3$ the asymptotic solution takes the finite value $x^*(r) = 1 - 1/r$ for almost any initial condition. Beyond $r = 3$ the asymptotic solution bifurcates, x_n oscillates between two values x_1^* and x_2^*, and the solution has period 2. Increasing r other bifurcations appear at sharp values. Very complex dynamic behaviour arises in the range $r \in [3.57, 1]$: the map has bands of chaotic behaviour, i.e., different initial conditions exponentially diverge, intertwined with windows of periodic behaviour.

A coupled logistic map lattice is a discrete array of coupled continuous variables, x_n^i, each of them evolving in time following (1). A typical interaction that we use here is a nearest neighbours spatial coupling of Laplacian form:

$$x_n^i = f(x_{n-1}^i) + \frac{v}{2}[f(x_{n-1}^{i-1}) - 2f(x_{n-1}^i) + f(x_{n-1}^{i+1})], \tag{2}$$

with $x_n^{i+N} = x_n^i$ for all n, with N being the number of elements on the ring. The initial condition is usually chosen to be random and thus taken from the uniform distribution on the interval $[0, 1]$ independently on each site. v is the coupling strength between the nodes and plays the role of a viscosity. In a nutshell, the dynamics of this system is characterised by a competition between the diffusion term, that tends to produce an homogeneous behaviour in space, and the chaotic motion of each unit, that favours spatial inhomogeneous behaviour due to the high sensitivity to the initial conditions.

The main idea is to take the continuum limit of the CML using the usual discretisation of time and space derivatives, e.g. $\partial h/\partial t \leftrightarrow (h_{n+1}^i - h_n^i)/\delta t$, etc. with δt the time-step and δx the lattice spacing that equal one in our system of units. The CML of logistic elements then becomes

$$\frac{\partial h}{\partial t} = \frac{vr}{2}(1 - 2h)\frac{\partial^2 h}{\partial x^2} - vr\left(\frac{\partial h}{\partial x}\right)^2 + (r - 1)h - rh^2, \tag{3}$$

where we called x the coordinate ($i\delta x \to x$), t the time ($n\delta t \to t$), and h the field $[x_n^i \to h(x, t) = h]$.

One immediately notices that Eq. (3) looks like a KPZ or FKPP equation but:

(i) By definition the field h is bounded and takes values in the unit interval. Thus, the resulting equation should have an effective confining potential that limits the field to a finite range. The field is not bounded in KPZ but it is in the FKPP equation.

(ii) The elastic term is here multiplied by a field-dependent viscosity

$$v(h) \equiv \frac{vr}{2}(1 - 2h). \tag{4}$$

First, $v(h)$ is negative for $h < \frac{1}{2}$ which implies an instability in the hydrodynamic limit. It was shown in Yakhot [8] and L'vov et al. [9] that in the Kuramoto–Sivashinsky equation a similar instability taps the system and so creates an effective 'noise' leading to a mapping onto the KPZ equation. The confining potential restraints the instabilities caused by $v(h) < 0$. Second, if h remains bounded the viscosity takes values on a finite interval. However, we expect the field-dependent bare viscosity to be renormalised at large scales by the effect of the non-linear terms (see below) and thus its precise value seems not to be very important.

(iii) The second, non-linear term is of the form of the one in the KPZ equation with a negative coupling $\lambda \equiv -vr$, though the sign of λ should not be important. This term does not exist in the FKPP equation.

(iv) The last two terms read

$$\eta(x, t) \equiv (r - 1)h(x, t) - rh^2(x, t). \tag{5}$$

We note that these terms are not present in the KPZ equation. In order to compare to the latter we argued that they have a double identity: on the one hand η behaves roughly as a short-range correlated noise in space and time; on the other hand it can be interpreted as a force derived from a confining potential

$$\eta = -\frac{\partial V(h)}{\partial h}, \quad V(h) = -\frac{(r - 1)}{2}h^2 + \frac{r}{3}h^3. \tag{6}$$