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Coupled continuous time random walks in finance

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Abstract

Continuous time random walks (CTRWs) are used in physics to model anomalous diffusion, by incorporating a random waiting time between particle jumps. In finance, the particle jumps are log-returns and the waiting times measure delay between transactions. These two random variables (log-return and waiting time) are typically not independent. For these coupled CTRW models, we can now compute the limiting stochastic process (just like Brownian motion is the limit of a simple random walk), even in the case of heavy-tailed (power-law) price jumps and/or waiting times. The probability density functions for this limit process solve fractional partial differential equations. In some cases, these equations can be explicitly solved to yield descriptions of long-term price changes, based on a high-resolution model of individual trades that includes the statistical dependence between waiting times and the subsequent log-returns. In the heavy-tailed case, this involves operator stable space–time random vectors that generalize the familiar stable models. In this paper, we will review the fundamental theory and present two applications with tick-by-tick stock and futures data.

Keywords: Anomalous diffusion; Continuous time random walks; Heavy tails; Fractional calculus

Continuous time random walk (CTRW) models impose a random waiting time between particle jumps. They are used in statistical physics to model anomalous diffusion, where a cloud of particles spreads at a rate different than the classical Brownian motion, and may exhibit skewness or heavy power-law tails. In the coupled model, the waiting time and the subsequent jump are dependent random variables. See Metzler and Klafter [1,2] for a recent survey. Continuous time random walks are closely connected with fractional calculus. In the classical random walk models, the scaling limit is a Brownian motion, and the limiting particle densities solve the diffusion equation. The connection between random walks, Brownian motion, and the diffusion equation is due to Bachelier [3] and Einstein [4]. Sokolov and Klafter [5] discuss modern extensions to include heavy-tailed jumps, random waiting times, and fractional diffusion equations.

In Econophysics, the CTRW model has been used to describe the movement of log-prices [6–10]. An empirical study of tick-by-tick trading data for General Electric stock during October 1999 (Fig. 1, left) in

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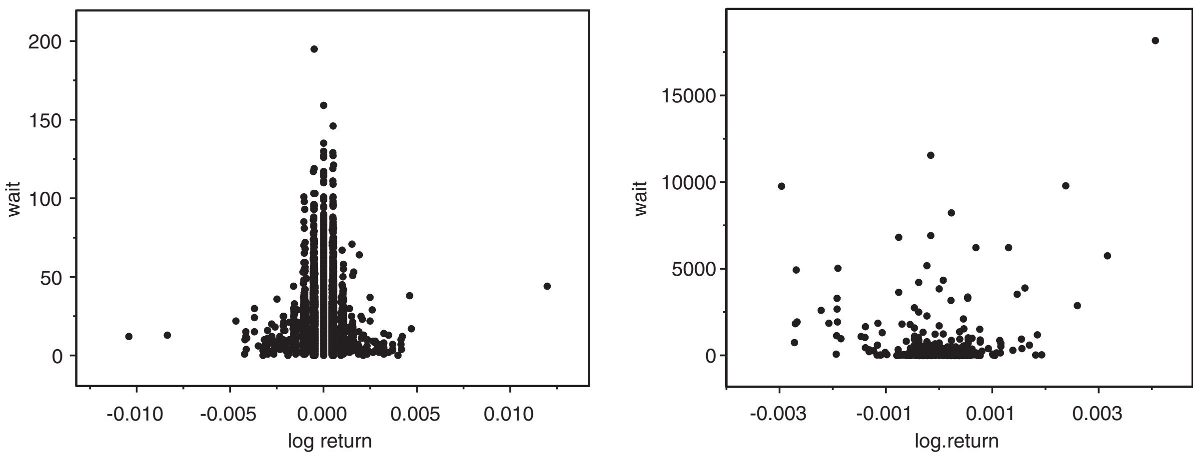

Fig. 1. Waiting times in seconds and log-returns for General Electric stock (left) and LIFFE bond futures (right) show significant statistical dependence.

Raberto et al. [8] uses a Chi-square test to show that the waiting times and the subsequent log-returns are not independent. These data show that long waiting times are followed by small (in absolute value) returns, while large returns follow very short waiting times. This dependence seems intuitive for stock prices, since trading accelerates during a time of high volatility [11]. LIFFE bond futures from September 1997 (Fig. 1, right) show a different behavior, where long waiting times go with large returns. See [8] for a detailed description of the data. In both cases, it seems clear that the two variables are dependent. In the remainder of this paper, we will describe how coupled continuous time random walks can be used to create a high-resolution model of stock prices in the presence of such dependence between waiting times and log-returns. We will also show how this fine scale model transitions to an anomalous diffusion limit at long time scales, and we will describe fractional governing equations that can be solved to obtain the probability densities of the limiting process, useful to characterize the natural variability in price in the long term.

Let $P(t)$ be the price of a financial issue at time t. Let $J_1, J_2, J_3, \ldots$ denote the waiting times between trades, assumed to be non-negative, IID random variables. Also let $Y_1, Y_2, Y_3, \ldots$ denote the log-returns, assumed to be IID. We specifically allow that J_i and Y_i are coupled, i.e., dependent random variables for each n. Now the sum $T_n = J_1 + \cdots + J_n$ represents the time of the nth trade. The log-returns are related to the price by $Y_n = \log[P(T_n)/P(T_{n-1})]$ and the log-price after n trades is $S_n = \log[P(T_n)] = Y_1 + \cdots + Y_n$. The number of trades by time $t>0$ is $N_t = \max\{n : T_n \leqslant t\}$, and the log-price at time t is $\log P(t) = S_{N_t} = Y_1 + \cdots + Y_{N_t}$.

The asymptotic theory of CTRW models describes the behavior of the long-time limit. For more details see [12–14]. The log-price $\log P(t) = S_{N_t}$ is mathematically a random walk subordinated to a renewal process. If the log-returns Y_i have finite variance then the random walk S_n is asymptotically normal. In particular, as the time scale $c \to \infty$ we have the stochastic process convergence $c^{-1/2} S_{[ct]} \Rightarrow A(t)$, a Brownian motion whose densities $p(x,t)$ solve the diffusion equation $\partial p/\partial t = D\partial^2 p/\partial x^2$ for some constant $D>0$ called the diffusivity. If the waiting times J_i between trades have a finite mean λ^{-1} then the renewal theorem [15] implies that $N_t \sim \lambda t$ as $t \to \infty$, so that $S_{N_t} \approx S_{\lambda t}$, and hence the CTRW scaling limit is still a Brownian motion whose densities solve the diffusion equation, with a diffusivity proportional to the trading rate λ. If the symmetric mean zero log-returns have power-law probability tails $P(|Y_i|>r) \approx r^{-\alpha}$ for some $0<\alpha<2$ then the random walk S_n is asymptotically α-stable, and $c^{-1/\alpha} S_{[ct]} \Rightarrow A(t)$ where the long-time limit process $A(t)$ is an α-stable Lévy motion whose densities $p(x,t)$ solve a (Riesz–Feller) fractional diffusion equation $\partial p/\partial t = D\partial^\alpha p/\partial |x|^\alpha$. If the waiting times have power-law probability tails $P(J_i>t) \approx t^{-\beta}$ for some $0<\beta<1$ then the random walk of trading times T_n is also asymptotically stable, with $c^{-1/\beta} T_{[ct]} \Rightarrow D(t)$ a β-stable Lévy motion. Since the number of trades N_t is inverse to the trading times (i.e., $N_t \geqslant n$ if and only if $T_n \leqslant t$), it follows that the renewal process is asymptotically inverse stable $c^{-\beta} N_{ct} \Rightarrow E(t)$ where $E(t)$ is the first passage time when $D(\tau)>t$. Then the log-price $\log P(t) = S_{N_t}$ has long-time asymptotics described by $c^{-\beta/\alpha} \log P(ct) \Rightarrow A(E_t)$, a subordinated process. If the waiting times J_i and the log-returns Y_i are uncoupled (independent) then the CTRW scaling limit process