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Lévy processes and Schrödinger equation

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ABSTRACT

We analyze the extension of the well known relation between Brownian motion and the Schrödinger equation to the family of the Lévy processes. We consider a Lévy–Schrödinger equation where the usual kinetic energy operator – the Laplacian – is generalized by means of a selfadjoint, pseudodifferential operator whose symbol is the logarithmic characteristic of an infinitely divisible law. The Lévy–Khintchin formula shows then how to write down this operator in an integro-differential form. When the underlying Lévy process is stable we recover as a particular case the fractional Schrödinger equation. A few examples are finally given and we find that there are physically relevant models – such as a form of the relativistic Schrödinger equation – that are in the domain of the non stable Lévy–Schrödinger equations.

1. Introduction

That the Schrödinger equation can be linked to some underlying stochastic process has been well known for a long time. This idea has received along the years a number of different formulations: from the Feynman path integral [1], through the Bohm–Vigier model [2], to the Nelson stochastic mechanics [3,4]. In all these models the underlying stochastic process powering the random fluctuations is a Gaussian Brownian motion, and the focus of interest is the (non relativistic) Schrödinger equation of quantum mechanics. This particular choice is understandable because on one hand the Gaussian Brownian motion is the the most natural and widely explored example of Markov process available, and on the other hand its connection with the Schrödinger equation has always lent the hope of a finer understanding of quantum mysteries.

In the framework of stochastic mechanics, however, this standpoint can be considerably broadened since in fact this theory is a model for systems more general than quantum mechanics: a *dynamical theory of Brownian motion* that can be applied to several physical problems [5–7]. On the other hand in recent years we have witnessed a considerable growth of interest in non Gaussian stochastic processes, and in particular in the Lévy processes [8–12]. This is a field that was initially explored in the 30's and 40's of last century [13–15], but that achieved a full blossoming of research only in the last decades, also as a consequence of the tumultuous development of computing facilities. This interest is witnessed by the large field of the possible applications of these more general processes from statistical mechanics [7] to mathematical finance [10,16, 17]. In the physical field, however, the research scope is presently rather confined to a particular kind of Lévy processes: the stable processes and the corresponding fractional calculus [18,19], while in the financial domain a vastly more general type of processes is at present in use. For instance also recently [20] the possibility of widening the perspective of the Schrödinger–Brownian pair has been considered, but that has been confined only to a fractional Schrödinger equation. The association of the more general Lévy infinitely divisible processes to the Schrödinger equation, instead, has been recognized

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as an important tool only in precious few papers [21,22] where the main interest of the authors was focused on the probabilistic interpretation of the relativistic quantum equations. Here, on the other hand, we suggest that the stochastic mechanics should be considered as a dynamical theory of the infinitely divisible processes showing time reversal invariance, and that the horizon of its applications should be widened even to cases different from quantum systems (see for example [5, 6]).

This paper is devoted to a discussion of a generalization of the Schrödinger equation which takes into account the entire family of the Lévy processes: we will propose an equation where the infinitesimal generator of the Brownian semigroup (the Laplacian) is substituted by the more general generator of a Lévy semigroup. As it happens this will be a pseudodifferential operator (as, in particular, in the fractional case), and the Lévy–Khintchin formula will give us the opportunity to write it down in the form of an explicit integro-differential operator by putting in evidence its continuous (Gaussian) and its jumping (non Gaussian) parts. It is important to recall indeed that all the non Gaussian Lévy processes are characterized by the fact that their trajectories make jumps: a feature that can help to explain particular physical phenomena, as for instance the halo formation in intense charged particle beams in the accelerators [5,6], and the relativistic quantum mechanics [21, 22]. The advantages of this formulation are many: first of all the widening of the increment laws from the stable to the infinitely divisible case will offer the possibility of having realistic, finite variances. As we will discuss later indeed, while all the non Gaussian, stable distributions have divergent variances — so that the range of the x decay rates of the stable probability density functions cannot exceed x^{-3} — this is not the case for the more general infinite divisible laws. On the other hand both the possible presence of a Gaussian component in the Lévy–Khintchin formula, and the wide spectrum of decay velocities of the increment probability densities will afford the possibility of having models with differences from the usual Brownian (and usual quantum mechanical, Schrödinger) case as small as we want. In this sense we could speak of small corrections to the quantum mechanical, Schrödinger equation. Last but not least, there are examples of non stable Lévy processes which are connected to a particular form of the quantum, relativistic Schrödinger equation: an important link that was missing in the original Nelson model. It seems in fact [21,22] that we can only recover some kind of relativistic quantum mechanics if we widen the field of the underlying stochastic processes at least to that of the selfdecomposable, jumping Lévy processes. To avoid formal complications we will confine our discussion to the case of processes in just one spatial dimension: generalizations will be straightforward.

2. A heuristic discussion

Let us start from the non relativistic, free Schrödinger equation associated with its propagator or Green function $G(x, t|y, s)$ (see for example Ref. [1])

$$\mathrm{i}\hbar\partial_t\psi(x,t) = -\frac{\hbar^2}{2m}\,\partial_x^2\psi(x,t) \tag{1}$$

$$G(x,t|y,s) = \frac{1}{\sqrt{2\pi\,\mathrm{i}(t-s)\hbar/m}}\,\mathrm{e}^{-\frac{(x-y)^2}{2\mathrm{i}(t-s)\hbar/m}} \tag{2}$$

$$\psi(x,t) = \int_{-\infty}^{+\infty} G(x,t|y,s)\,\psi(y,s)\mathrm{d}y \tag{3}$$

and compare it with the Fokker–Planck equation of a Wiener process (Brownian motion) with diffusion coefficient D, *pdf* (probability density function) $q(x, t)$ and transition *pdf* $p(x, t|y, s)$ (see for example Ref. [23])

$$\partial_t q(x,t) = D\,\partial_x^2 q(x,t) \tag{4}$$

$$p(x,t|y,s) = \frac{1}{\sqrt{4\pi(t-s)D}}\,\mathrm{e}^{-\frac{(x-y)^2}{4(t-s)D}} \tag{5}$$

$$q(x,t) = \int_{-\infty}^{+\infty} p(x,t|y,s)q(y,s)\mathrm{d}y. \tag{6}$$

It is apparent that there is a simple, formal procedure transforming the two structures one into the other:

$$D = \frac{\hbar}{2m}, \quad t \longleftrightarrow \mathrm{i}t$$

It is well known that this is just the result of a time analytic continuation in the complex plane. There are of course important differences between G and p. For example while p and q are well behaved *pdf*'s, G is not a wave function, as can be seen also from a simple dimensional argument. This simple symmetry can then be deceptive, and a better understanding of its true meaning can be achieved either by means of the Feynman path integration with a free Lagrangian of the usual quadratic form, or through the Madelung decomposition [24] of (1) and its subsequent stochastic mechanical model [3,4]. Either way produces in the end the well known association between the Schrödinger equation and a very special form of background