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Complex traveling wave solutions to the Fisher equation ☆

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Abstract

There is the widespread existence of wave phenomena in physics, chemistry and biology. This clearly necessitates a study of traveling waves in depth and of the modeling and analysis involved. In the present paper, we study the Fisher equation by means of the first integral method, which is based on the ring theory of commutative algebra. A traveling wave solution is obtained, which indicates that the analytical solutions in the literature are particular cases of our result.

Keywords: Traveling waves; First integral; Fisher equation; Autonomous system

1. Introduction

The problems of the propagation of nonlinear waves have fascinated scientists for over two hundred years. The modern theory of nonlinear waves, like many areas of mathematics, had its beginnings in attempts to solve specific problems, the hardest among them being the propagation of waves in water. There was significant activity on this problem in the 19th century and the beginning of the 20th century, including the classic work of Stokes, Lord Rayleigh, Korteweg and de Vries, Boussinesque, Benard and Fisher to name some of the better remembered examples [1,2]. One particularly noteworthy contribution was the explosion of activity unleashed by the numerical discovery of the soliton by Zabusky and Kruskal in the early sixties, and the earliest theoretical explanation by Gardner, Greene, Kruskal, and Miura in the later part of that decade, which subsequently led to the present-day theory of integrable partial differential equations. Nonlinear waves and coherent structures is an interdisciplinary area that has many important applications, including nonlinear optics, hydrodynamics, plasmas and solid-state physics. In fact, for any physical system where the dynamics is driven by, and mainly determined by, phase coherence of the individual waves, it has applications and consequences.

Modern theories describe nonlinear waves and coherent structures in a diverse variety of fields, including general relativity, high energy particle physics, plasmas, atmosphere and oceans, animal dispersal, random media, chemical reactions, biology, nonlinear electrical circuits, and nonlinear optics. For example, in the

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latter, the mathematics developed for describing the propagation of information via optical solitons is most striking, attaining an incredible accuracy. It has been experimentally verified and spans twelve orders of magnitude: from the wavelength of light to transoceanic distances. It also guides the practical applications in modern telecommunications. Many other nonlinear wave theories mentioned above can claim similar success.

Nowadays it has been universally acknowledged in the physical, chemical and biological communities that the reaction–diffusion equation plays an important role in dissipative dynamical systems. Typical examples are provided by the fact that there are many phenomena in mechanical engineering and biology where a key element or precursor of a developmental process seems to be the appearance of a traveling wave of chemical concentration (or mechanical deformation). When reaction kinetics and diffusion are coupled, traveling waves of chemical concentration can effect a biochemical change much faster than straight diffusional processes. This usually gives rise to reaction–diffusion equations which in one-dimensional space can look like

$$\frac{\partial u}{\partial t} = k_0 \frac{\partial^2 u}{\partial x^2} + f(u) \tag{1}$$

for a chemical concentration u, where k_0 is the diffusion coefficient, and $f(u)$ represents the kinetics.

When $f(u)$ is linear, i.e., $f(u) = k_2 u + k_1$, where both k_1 and k_2 are constants, then in many instances the Eq. (1) can be solved by the separation of variables methods. However if, as in many of the applications considered in Ref. [3], $f(u)$ is nonlinear, then the problem is much more intractable. Indeed, it is not usually possible to obtain general exact analytical traveling wave solutions and one must analyze such problems numerically [4]. Despite this, however, under some particular circumstances, many nonlinear evolutionary equations have traveling wave solutions of special types, which are of fundamental importance to our understanding of biological phenomena modeling evolutionary equations.

The classic and simplest case of the nonlinear reaction–diffusion equation is when $f(u) = \alpha u(1-u)$, i.e.,

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \alpha u(1-u), \tag{2}$$

which is the so-called Fisher equation. It was suggested by Fisher as a deterministic version of a stochastic model for the spatial spread of a favored gene in a population [5]. (Although this equation is now referred to as the Fisher equation, the discovery, investigation and analysis of traveling waves in chemical reactions was first presented by Luther at a conference [6]. There, he stated that the wave speed is a simple consequence of the differential equations. This recently re-discovered paper has been translated by Arnold et al. [7] and Luther's remarkable discovery and analysis of chemical waves has been put in a modern context by Showalter and Tyson [8]). In the last century, the Fisher equation has became the basis for a variety of models for spatial spread, for example, in logistic population growth models [9,10], flame propagation [11,12], neurophysiology [13], autocatalytic chemical reactions [14–16], branching Brownian motion processes [17], gene-culture waves of advance [18], the spread of early farming in Europe [19,20], and nuclear reactor theory [21]. It is incorporated as an important constituent of nonscalar models describing excitable media, e.g., the Belousov–Zhabotinsky reaction [22]. In chemical media, the function $u(t,x)$ is the concentration of the reactant and the positive constant α represents the rate of the chemical reaction. In media of other natures, u might be temperature or electric potential. The medium described by (2) is often referred to as a bistable medium because it has two homogeneous stationary states, $u = 0$ and $u = 1$. A kink-like traveling wave solution of (2) describes a constant-velocity front of transition from one homogeneous state to another.

Finding solutions of nonlinear models is a difficult and challenging task. Several analytical methods have been developed for obtaining wave solutions for pure dispersive nonlinear systems in one spatial dimension: the inverse scattering transfer [23], the Hirota method [24], Painlevé analysis [25], Cole–Hopf transform [26], Lamb's ansatz [27,28], etc. Some of these methods may be extended for $(2+1)$-dimensional systems (two spatial dimensions and one temporal variable). The problem of obtaining solutions for systems including dissipative losses turned out to be more complex. Even for the $(1+1)$ case, most of the above-mentioned methods do not work. Hence, the usual way of treating the problem is by qualitative analysis or numerical investigation. The seminal and now classical references for traveling waves of the Fisher equation are that by Kolmogorov, et al. [29], Mckean [30], Albowitz and Zeppetella [25], Fife [31] and Britten [10]. In Ref. [29], Kolmogorov et al. showed that any initial concentration which is one for large negative spatial variable x and