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The estimates of correlations in two-dimensional Ising model

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ABSTRACT

We investigate the correlation inequalities and the decay of correlations of stochastic Ising model in a rectangle with side length $2L \times K(L \ln L)^{1/2}$, where K is some positive constant. With different boundary conditions, at inverse temperature $\beta > \beta_c$ or $\beta < \beta_c$ and zero external field, we show some estimates of the correlation functions for the two-dimensional Ising model.

1. Introduction

In this paper, we consider the two-dimensional lattice Ising model (see Refs. [1–4]) on the rectangle $\Lambda_{L,M}$, where $\Lambda_{L,M} = [-L, L] \times [-M, M] \subset \mathbb{Z}^2$ and $M = K(L \ln L)^{1/2}$, K is some positive constant. With some different boundary conditions, at inverse temperature $\beta > \beta_c$ or $\beta < \beta_c$ and zero external field, the statistical properties of the correlation functions for the two-dimensional Ising model are investigated. The spin of the Ising model takes the spin value $+1$ or spin value -1, and it flips between the two orientations. At sufficiently low temperatures, we have known that the model exhibits phase transition, i.e., there is a critical point $\beta_c > 0$, if $\beta > \beta_c$, the Ising model exhibits phase transition. Correlations are related to the phase transition and the spin fluctuations of the model. As β increases (from 0), the correlations begin to extend, these correlations take the form of spin fluctuations, which are islands of a few spins each that mostly point in the same direction. As β approaches the critical inverse temperature β_c from below, spin fluctuations are present at all scales of length. At $\beta = \beta_c$, the correlations decay by a power law, but for $\beta > \beta_c$, there are two distinct pure phases. Correlations play an important role in studying the fluctuations of the phase interfaces for the statistical physics model, see Refs. [2, 4–10]. When $\beta \gg \beta_c$, by applying the theory of the cluster expansion, the fluctuations of the interfaces between the plus and minus phases can be controlled, from the normal ones occurring on scale $L^{\frac{1}{2}}$ to the large ones on scale L, see Ref. [2]. For β close to the critical value β_c, we are still unable to have such a result. So, for $\beta > \beta_c$, we estimate the correlations of the Ising model in a rectangle with side length $2L \times K(L \ln L)^{1/2}$ in the present paper, and the statistical properties of interfaces of the Ising model are also discussed in Theorem 3.

Let $\mathbb{Z}^2$ be a square lattice, an element u of which is a pair of integers (u_1, u_2). For any finite subset $\Lambda \subset \mathbb{Z}^2$, let $\Omega_\Lambda = \{-1, +1\}^\Lambda$ denote the space of spin configurations on Λ, an element of Ω_Λ usually denoted $\xi_\Lambda = \{\xi(u) : u \in \Lambda\}$,

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whenever confusion does not arise, we will also omit the subscript Λ in the notation ξ_Λ. We consider a standard Ising model with the following system of Hamiltonians

$$H^\eta_\Lambda(\xi) = -\frac{1}{2}\sum_{\substack{u,v\in\Lambda\\|u-v|=1}} \xi(u)\xi(v) - \sum_{\substack{u\in\Lambda,v\notin\Lambda\\|u-v|=1}} \xi(u)\eta(v) \tag{1}$$

for every $\xi \in \Omega_\Lambda$, where $|\cdot|$ stands for the Euclidean distance and $\eta \in \{-1, 0, +1\}^{\mathbb{Z}^2}$ denotes the boundary condition. If we set $\eta(u) = +1$ for all $u \in \mathbb{Z}^2$, the boundary condition is called the plus boundary condition, if $\eta(u) = -1$ for all u, then the resulting boundary condition is called the minus boundary condition, and if $\eta(u) = 0$ for all u, then we call the resulting boundary condition the free boundary condition. The corresponding Hamiltonians are denoted by H^+_Λ, H^-_Λ and $H^\emptyset_\Lambda$ respectively. The finite Gibbs state $\mu^{\beta,\eta}_\Lambda$ at inverse temperature β is a probability measure on Ω_Λ given by

$$\mu^{\beta,\eta}_\Lambda(\xi) = [Z^{\beta,\eta}_\Lambda]^{-1}\exp[-\beta H^\eta_\Lambda(\xi)]$$

where $Z^{\beta,\eta}_\Lambda$ is called the partition function and is given by $Z^{\beta,\eta}_\Lambda = \sum_{\xi\in\Omega_\Lambda}\exp[-\beta H^\eta_\Lambda(\xi)]$.

The interesting case is that β is greater than the critical value β_c. In this case, the Gibbs measures μ^+_Λ and μ^-_Λ (corresponding to + and − boundary conditions respectively) will converge to different limits μ^+ and μ^- as Λ expands to the whole plane $\mathbb{Z}^2$, see Refs. [1,3,4]. A stochastic Ising model on Λ with boundary condition η is a continuous time Markov chain on Ω_Λ, whose generator is of the following form

$$(A^{\beta,\eta}_\Lambda f)(\xi) = \sum_{u\in\Lambda} c^\eta(u,\xi)[f(\xi^u) - f(\xi)] \tag{2}$$

acting on $L^2(\Omega_\Lambda, d\mu^{\beta,\eta}_\Lambda)$, where $\xi^u(v) = +\xi(v)$, if $v \neq u$ and $\xi^u(v) = -\xi(v)$, if $v = u$. $c^\eta(u,\xi)$ is the transition rates for the process, satisfying nearest neighbor interactions, attractivity, boundedness and detailed balance condition $c(u,\xi)^\eta\mu^{\beta,\eta}_\Lambda(\xi) = c(u,\xi^u)^\eta\mu^{\beta,\eta}_\Lambda(\xi^u)$, see [1].

2. Notations and definitions

Let $\mathbb{Z}^2_*$ be the dual lattice of $\mathbb{Z}^2$, i.e., $\mathbb{Z}^2_* = \mathbb{Z}^2 + (1/2, 1/2)$. For $u, v \in \mathbb{R}^2$, let $[u, v]$ be the closed segment with u, v as its endpoints. The edges of $\mathbb{Z}^2(\mathbb{Z}^2_*)$ are those $e = [u, v]$ with u, v nearest neighbors in $\mathbb{Z}^2(\mathbb{Z}^2_*)$. Given an edge e of $\mathbb{Z}^2$, e^* is the unique edge in $\mathbb{Z}^2_*$ that intersects e. We denote by $\mathbb{B}_\Lambda$ the set of edges such that both the endpoints are in Λ and by $\bar{\mathbb{B}}_\Lambda$ the set of all edges with at least one endpoint in Λ. Given $\Lambda \subset \mathbb{Z}^2$, we let $\Lambda^c = \mathbb{Z}^2 \setminus \Lambda$ and define Λ^* as the set of all $u \in \mathbb{Z}^2_*$ such that $d(u,\Lambda) = \frac{1}{\sqrt{2}}$, where $d(u,\Lambda) = \inf\{|u - v| : v \in \Lambda\}$. The set of the dual edges is defined as $\mathbb{B}^*_\Lambda = \{e^* : e \in \bar{\mathbb{B}}_\Lambda\}$. The interior and exterior boundaries of Λ are defined by

$$\partial_{\text{int}}\Lambda \equiv \{u \in \Lambda : \exists v \notin \Lambda, |u - v| = 1\}, \qquad \partial_{ext}\Lambda \equiv \{u \notin \Lambda : \exists v \in \Lambda, |u - v| = 1\}$$

and $\partial_{\text{int}}\Lambda^*$, $\partial_{ext}\Lambda^*$ are defined in a similar way.

For simplicity, we call an edge in $\mathbb{Z}^2_*$ a bond, so that we can distinguish it from the edges in $\mathbb{Z}^2$. We say that a neighboring pair u and v in $\mathbb{Z}^2$ are separated by a bond e^* if the edge $e = [u, v]$ intersects e^*. Let $\Lambda \subset \mathbb{Z}^2$ and $\eta \in \{-1, 0, +1\}^{\mathbb{Z}^2}$ be fixed, for every configuration $\xi \in \Omega_\Lambda$, we denote by $\Gamma(\xi)$ the collection of all bonds separating neighboring sites u and v such that: (i) $u, v \in \Lambda$, and $\xi(u)\xi(v) = -1$ or (ii) $u \in \Lambda$, $v \in \partial_{ext}\Lambda$ and $\xi(u)\eta(v) = -1$. We divide $\Gamma(\xi)$ into connected components. Further we use the convention that any pair of orthogonal bonds that intersect in a given site u^* of the dual lattice $\mathbb{Z}^2_*$ are a linked pair of bonds iff they are both on the same side of the forty-five degrees line across u^*, then we regard that two linked pairs at u^* are not connected at u^*. By this convention, each connected component of $\Gamma(\xi)$, say Γ, has the following properties: (1) if $u^* \in \Lambda^* \setminus \partial_{\text{int}}\Lambda^*$, then the number of bonds in Γ that intersect u^* is always even; (2) bonds in Γ can be ordered as $e^*_0, e^*_1, \ldots, e^*_n$, so that e^*_i and e^*_{i+1} have a common vertex for every i, and if Γ has a point u^* at which 4 bonds in Γ which intersect u^*, then there are $i \neq j$ such that these 4 bonds are divided into two linked pairs $\{e^*_i, e^*_{i+1}\}$ and $\{e^*_j, e^*_{j+1}\}$. We call these components of $\Gamma(\xi)$ contours in ξ (with boundary condition η). If for any $u^* \in \mathbb{Z}^2_*$, the number of bonds in the contour Γ which intersect u^* is even, then we call Γ a closed contour. A contour which is not closed is called an open contour. The length $|\Gamma|$ of a contour is simply the number of bonds in Γ.

When Λ is a rectangle $\Lambda = \Lambda_{L,M}$, we define $[h]$ boundary condition by

$$[h](u) = \begin{cases} -1, & \text{if } u_2 \geq M - h + 1 \\ +1, & \text{if } u_2 \leq M - h \end{cases} \tag{3}$$

for $u = (u_1, u_2) \in \mathbb{Z}^2$. So in particular, [0] boundary condition means −1 on the top side of the rectangle and +1 on the remaining three sides. Let $\Gamma^{[h]}_\Lambda(\xi)$ denote the unique open contour produced by a configuration ξ.

Now we give the notations and the fundamental results of the surface tension, for the details see Refs. [2,8]. We denote by $\tau_\beta(\theta)$ the surface tension at angle θ, which measures the free energy of an interface in the direction orthogonal to the