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Random walk, cluster growth, and the morphology of urban conglomerations

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Abstract

We propose a new model of cluster growth according to which the probability that a new unit is placed in a point at a distance r from the city center is a Gaussian with mean equal to the cluster radius and variance proportional to the mean, modulated by the local density $\rho(r)$. The model is analytically solvable in $d = 2$ dimensions, where the density profile varies as a complementary error function. The model reproduces experimental observations relative to the morphology of cities, determined via an original analysis of digital maps with a very high spatial resolution, and helps understanding the emergence of vehicular traffic.

Keywords: Growth processes; Diffusion-limited aggregation; Traffic

1. Introduction

Important problems related to cluster growth processes occur in a number of different disciplines, ranging from physics to biology and transportation engineering, and several microscopic models have been proposed to describe the growth of both compact clusters, like crystals or tumors [1,2], and fractal clusters, like colloidal aggregates or snowflakes [3]. All of these models are characterized by the presence of an 'active' zone on the surface of the cluster where the cluster growth takes place. Despite the simplicity which characterizes the microscopic dynamics of these cluster growth models, analytical solutions for the temporal evolution and for the spatial dependency of cluster properties, like the density profile or the width of the active zone, are difficult to obtain. To this end one usually resorts to models for the evolution of the cluster surface, such as the Kardar–Parisi–Zhang model [4], or to extensive numerical simulations [5–7].

These numerical simulations have suggested that, at least in the case of the Eden Model and of the DLA model, the radially averaged probability $P(r,N)\,dr$ that the $(N+1)$th cluster unit is deposited within a shell of width dr at a distance r from the center of mass of the cluster is well approximated for r and N large by a

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Gaussian distribution,

$$P(r,N) = \frac{1}{(2\pi)^{1/2}\sigma_N} \exp\left[-\frac{(r-r_N)^2}{2\sigma_N^2}\right] , \tag{1}$$

with mean $r_N \propto N^{\nu}$ and variance $\sigma \propto N^{\nu'}$; the scaling exponents ν and ν' are model dependent. This growth probability distribution leads to a density profile $\rho(r,N)$ of a cluster of size N in d dimensions,

$$\rho(r,N) = \frac{1}{S_d r^{d-1}} \int_0^N P(r,N')\,\mathrm{d}N' , \tag{2}$$

where S_d is the surface area of the d dimensional unit sphere ($S_2 = 2\pi$, $S_3 = \pi$), which can be evaluated in the limit $N \to \infty$ and r large [5]: $\rho(r, N=\infty) \propto r^{-d+1/\nu}$.

Starting from Eq. (1) and from some considerations about the asymptotic behavior of $P(r,N)$ in the $r \to 0$ and $r \to \infty$ limits, in this paper we elaborate a new model for the growth of compact clusters, the random walk growth model (RWG). This is based on the simple idea that the cluster radius grows as a random walker subject to a drift, which gives a growing probability distribution $P_{\mathrm{RWG}}(r,N) \propto r^{d-1}P(r,N)$. We solve the model in $d = 2$ dimensions, showing that the density profile varies as a complementary error function. As an application of the proposed model we have studied the morphology of several European cities, which are growing clusters. Via an original analysis of digital maps with a very high resolution [8] we have determined the spatial dependence of their density of streets $\rho_s(r)$, which appears to be very well described by the RWG model.

2. The random walk cluster growth model

In a large number of cluster growth models (DLA, Eden, Solid-on-solid, Random Deposition, ...), a cluster grows as a new cluster unit is placed near an existing one. Therefore, in order for a cluster to grow in a given location $\vec{r}$, at least a cluster unit must be present near $\vec{r}$. In this respect, it is surprising that the growth probability of Eq. (1) depends on the cluster density $\rho(r,N)$ only through r_N and σ_N; instead, one would have expected the radially averaged probability of placing a cluster unit in a shell at a distance r from the cluster center to be proportional to the number of cluster units which occupy the shell, i.e., $P(r,N) \propto \rho(r,N)^{d-1} \propto r^{D_f-1}$, where $D_f = 1/\nu$ is the fractal dimension of the cluster.

We want also to point out that if $P(r,N) \propto r^{D_f-1}$, then Eq. (2) predicts the cluster density to diverge as $\rho(r) \propto r^{D_f-d}$ when $r \to 0$. We therefore expect a crossover in $P(r,N)$ which must be proportional to r^{d-1} when $r \to 0$, and proportional to r^{D_f-1} when $r \to \infty$.

Inspired by Eq. (1) and keeping in mind the above considerations, here we define a new model for the growth of compact clusters, where no crossover in the radial growing probability is expected as $D_f = d$, which gives rise to a growing probability distribution $P_{\mathrm{RWG}}(r,N) \propto r^{d-1}P(r,N)$. The model is defined by assuming (1) that the cluster mass N is related to the mean cluster radius r_N by $N = (r_N/r_0)^d$ in d spatial dimensions; and (2) that the radius of the cluster evolves as a random walker subject to a drift: at each updating step the radius varies of a quantity taken from a distribution with mean $v > 0$ (drift velocity) and variance σ^2.

Under these assumptions the radially averaged probability $P_{\mathrm{RWG}}(r,N)\,\mathrm{d}r$ that the $(N+1)$th cluster unit is deposited within a shell of width $\mathrm{d}r$ at a distance r from the center of mass of the cluster is

$$P_{\mathrm{RWG}}(r,N) = \frac{r^k}{\mu_N^{(k)}} P(r,N) , \tag{3}$$

where $P(r,N)$ is given in Eq. (1), $\mu_N^k = \int_0^\infty \mathrm{d}r\, r^k P(r,N)$ and we assume $k = d-1$ as discussed above. The growth dynamics sets $\nu = 1/d$ and $\nu' = \nu/2 = 1/2d$. Note that for $k < d-1$ the cluster density diverges when $r \to 0$, while for $k > d-1$ it does not decrease monotonically, and has a maximum at $r > 0$.

We restrict our analysis to $d = 2$ dimensions, which is the more interesting case from a physical viewpoint, but an analytical treatment is also possible for $d = 3$.