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Physica A 359 (2006) 177–212

PHYSICA A

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Coarse-grained distributions and superstatistics

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Received 21 September 2004; received in revised form 28 May 2005
Available online 14 July 2005

Abstract

We show an interesting connection between non-standard (non-Boltzmannian) distribution functions arising in the theory of violent relaxation for collisionless stellar systems [D. Lynden-Bell, Mon. Not. R. Astron. Soc. 136 (1967) 101.] and the notion of superstatistics recently introduced by [Beck and Cohen Physica A 322 (2003) 267]. The common link between these two theories is the emergence of coarse-grained distributions arising out of fine-grained distributions. The coarse-grained distribution functions are written as a superposition of Boltzmann factors weighted by a non-universal function. Even more general distributions can arise in case of incomplete violent relaxation (non-ergodicity). They are stable stationary solutions of the Vlasov equation. We also discuss analogies and differences between the statistical equilibrium state of a multi-components self-gravitating system and the metaequilibrium (or quasi-equilibrium) states of a collisionless stellar system. Finally, we stress the important distinction between entropies, generalized entropies, relative entropies and H-functions. We discuss applications of these ideas in two-dimensional turbulence and for other systems with long-range interactions.

Keywords: Violent relaxation; Superstatistics; Generalized entropies; H-functions

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0378-4371/$ - see front matter
doi:10.1016/j.physa.2005.06.043

1. Introduction

Recently, several researchers have questioned the "universality" of the Boltzmann distribution in physics. This problem goes back to Einstein himself who did not accept Boltzmann's principle $S = k \ln W$ on a general scope because he argued that the statistics of a system (W) should follow from its dynamics and cannot have a universal expression [1,2]. In 1988, Tsallis introduced a generalized form of entropy in an attempt to describe complex systems [3]. This was the starting point for several generalizations of thermodynamics, statistical mechanics and kinetic theories (see, e.g., Ref. [4]). A lot of experimental and numerical studies (in an impressive number of domains of physics) have then shown that complex systems exhibit non-standard distributions and that, in many cases, they can be fitted by Tsallis q-distributions [5]. However, there also exists physical systems (like those that we shall consider here) that are described neither by Boltzmann nor by Tsallis distributions.

An important question is to understand *why* non-standard distributions and generalized entropies emerge in a system. We have argued that non-standard distributions arise when microscopic constraints are in action [6]. They sometimes appear as "hidden constraints" inaccessible to the observer. For "simple systems", the energetically accessible microstates are *equiprobable* and a standard combinatorial analysis leads to the Boltzmann entropy. Then, the equilibrium distribution (most probable macrostate) maximizes the Boltzmann entropy at fixed macroscopic constraints (mass, energy,...). For "complex systems", the a priori accessible microstates are *not* equiprobable, some being even forbidden, contrary to what is postulated in ordinary statistical mechanics. The non-equiprobability of microstates can be due to microscopic constraints (of various origin) that affect the dynamics. In certain cases, the microscopic constraints can be dealt with by using a generalized form of entropy. In principle, this entropy $S = \ln W'$ should be obtained from a counting analysis by assuming that the microstates which satisfy the macroscopic constraints *and the microscopic constraints* are equiprobable. An example of microscopic constraints is provided by the Pauli exclusion principle in quantum mechanics which prevents two fermions with the same spin to occupy the same site in phase space. Because of this constraint, the Boltzmann entropy is replaced by the Fermi–Dirac entropy which puts a bound $f(\mathbf{x}, \mathbf{v}) \leqslant \eta_0$ on the maximum value of the distribution function. In this example, the exclusion principle is explained by quantum mechanics so it has a fundamental origin. Another example is when the particles are subject to an excluded volume constraint. In simplest models (e.g., a lattice model), this is accounted for by introducing a Fermi–Dirac type entropy in physical space which puts a bound $\rho(\mathbf{x}) \leqslant \sigma_0$ on the maximum value of the spatial density. These entropies can be obtained from a combinatorial analysis which carefully takes into account the fact that two particles cannot be in the same microcell in phase space or in physical space. More generally, we can imagine other situations where some microscopic constraints (not necessarily of fundamental origin) act on the system and lead to non-standard forms of distribution functions and entropies.