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Info-quantifiers' map-characterization revisited

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A B S T R A C T

We highlight the potentiality of a special Information Theory (IT) approach in order to unravel the intricacies of nonlinear dynamics, the methodology being illustrated with reference to the logistic map. A rather surprising *dynamic feature* $\rightarrow$ *plane-topography* map becomes available.

1. Introduction

A great ideal of interest exists in the behavior of dynamical systems that are highly sensitive to initial conditions, a sensitivity popularly referred to as the butterfly effect. Small differences in initial conditions (such as those due to rounding errors in numerical computation) yield widely diverging outcomes for chaotic systems, rendering long-term prediction impossible in general [1]. This happens even though these systems are deterministic and their "future" is fully determined by their initial conditions, with no random elements involved, so that the deterministic nature of these systems does not make them predictable. Explanation of such behavior may be sought in various ways, an important one being the through analysis of nonlinear models, that usually yields a wealth of interesting information.

Information theory, in turn, is a powerful weapon in the theoretical physicist's arsenal [2] that has been put to good use in the analysis just referred to, with an exuberant literature dealing with such matters (see references given below in the text). The present effort tries to add, hard as it sounds, some new ingredients to this kind of approach, and we expect to convince the readers in the forthcoming sections that this is not an empty claim.

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2. Shannon entropy & Fisher Information Measure

Given a continuous probability distribution function (PDF) $f(x)$, its *Shannon entropy S* is [3]

$$S[f] = -\int f \ln(f) \mathrm{d}x, \tag{1}$$

a measure of "global character" that it is not too sensitive to strong changes in the distribution taking place on a small-sized region.

Such is not the case with *Fisher's information measure (FIM) F* [2,4], which constitutes a measure of the gradient content of the distribution f, thus being quite sensitive even to tiny localized perturbations. It reads [2]

$$F[f] = \int \frac{|\vec{\nabla} f|^2}{f} \mathrm{d}x. \tag{2}$$

FIM can be variously interpreted as a measure of the ability to estimate a parameter, as the amount of information that can be extracted from a set of measurements, and also as a measure of the state of disorder of a system or phenomenon [2,5]. Its most important property is the so-called Cramer–Rao bound, that we recapitulate in one-dimension, for simplicity's sake. The classical Fisher information associated with translations of a one-dimensional observable x with corresponding probability density $f(x)$ is [6]

$$I_x = \int \mathrm{d}x f(x) \left(\frac{\partial \ln f(x)}{\partial x} \right)^2, \tag{3}$$

which obeys the above referred to Cramer–Rao inequality

$$(\Delta x)^2 \geq I_x^{-1} \tag{4}$$

involving the variance of the stochastic variable x [6]

$$(\Delta x)^2 = \langle x^2 \rangle - \langle x \rangle^2 = \int \mathrm{d}x f(x) x^2 - \left(\int \mathrm{d}x f(x) x \right)^2. \tag{5}$$

We insist in remarking that the gradient operator significantly influences the contribution of minute local f-variations to FIM's value, so that the quantifier is called a "local" one. Note that Shannon's entropy decreases with skewed distribution, while Fisher's information increases in such a case. Local sensitivity is useful in scenarios whose description necessitates appeal to a notion of "order" (see below).

Let now $P = \{p_i;\ i = 1, \ldots, N\}$ be a discrete probability distribution set, with N the number of possible states of the system under study. The concomitant problem of loss of information due to the discretization has been thoroughly studied (see, for instance, [7–9] and references therein) and, in particular, it entails the loss of FIM's shift-invariance, which is of no importance for our present purposes. In the discrete case, Shannon's quantifier is evaluated via

$$S[P] = -\sum_{i=1}^{N} p_i \ln(p_i), \tag{6}$$

and we define a "normalized" Shannon entropy as $H[P] = S[P]/S_{\max}$, where the denominator obtains for a uniform probability distribution.

For the FIM-computation measure, we follow the proposal of Ferri and coworkers [10] (among others)

$$F[P] = \frac{1}{4} \sum_{i=1}^{N-1} 2 \frac{(p_{i+1} - p_i)^2}{(p_{i+1} + p_i)}. \tag{7}$$

If our system is in a very ordered state and thus is represented by a very narrow PDF, we have a Shannon entropy $S \sim 0$ and a Fisher's information measure $F \sim F_{\max}$. On the other hand, when the system under study lies in a very disordered state one gets an almost flat PDF and $S \sim S_{\max}$ while $F \sim 0$. Of course, $S_{\max}$ and $F_{\max}$ are, respectively, the maximum values for the Shannon entropy and Fisher information measure. One can state that the general behavior of the Fisher information measure is opposite to that of the Shannon entropy [11].

3. Temporal information and methodologies for getting the pertinent PDFs

Information measures are functionals of a probability distribution function (PDF). In evaluating them, one has to properly determine this underlying PDF P (here associated with a given dynamical system or time series). This is an often neglected issue that indeed deserves detailed consideration. Why? Because P and the sample-space Ω are inextricably linked. Many schemes have been proposed for an adequate selection of pair (Ω, P). We can mention, among others: (a) procedures based