Physica A 389 (2010) 3081–3093

Contents lists available at ScienceDirect

Physica A

journal homepage: www.elsevier.com/locate/physa

Logistic equation of arbitrary order

Franciszek Grabowski

Rzeszów University of Technology, Department of Distributed Systems, W. Pola 2, 35-959 Rzeszów, Poland

ARTICLE INFO

Article history:
Received 14 September 2009
Received in revised form 10 February 2010
Available online 7 April 2010

Keywords:
Logistic equation
Arbitrary order
Feedback
Self-organization
Matching point

ABSTRACT

The paper is concerned with the new logistic equation of arbitrary order which describes the performance of complex executive systems X vs. number of tasks N, operating at limited resources K, at non-extensive, heterogeneous self-organization processes characterized by parameter f. In contrast to the classical logistic equation which exclusively relates to the special case of sub-extensive homogeneous self-organization processes at $f = 1$, the proposed model concerns both homogeneous and heterogeneous processes in sub-extensive and super-extensive areas. The parameter of arbitrary order f, where $-\infty < f < +\infty$, depends on both the coefficient of external resource utilization $u = N/K$, where $0 < u < 1$, and the internal microscopic character of realized processes related to the depth of feedback β. The coefficient β directly influences self-organization of processes by the change of microscopic parameters V_i, S_i, i and Z, where V_i is the number of references (visit) to the ith component of the system during the service of each task, S_i is the time of serving the task by the ith component, and Z is the think time of a given process. In the general case of complex system, parameters V_i, S_i, i and Z can have values in the range from 0 to $+\infty$. In this way the new equation includes all possible cases of a complex executive system's operation. Furthermore, it allows us to define the optimal matching point between X and N with f as the parameter. It also helps to balance the load in complex systems and to equip artificial systems with self-optimization mechanisms similar to those observed in natural systems.

1. Introduction

Natural and artificial systems, e.g. humans, animals, plants, ecosystems, markets, and ad hoc sensor wireless computer networks, depending on internal and external conditions, can stay in one of two diametrically different non-extensive thermodynamic states: sub-extensive or super-extensive [1]. The sub-extensive state means uncritical (reversible) self-organization (self-regulation) [2–9]. However, after reaching the percolation threshold [10–14], systems inevitably head for super-extensive critical (irreversible) catastrophic self-organization [15–26]. In this paper we show that the mechanisms of self-organization can move from the initial ideal equilibrium state described by the Malthus equation, to one of the two non-extensive states: uncritical or critical.

Ashby was the first who indicated two fundamental meanings of self-organization [2,3]. The first of them refers to "*the system that starts with its parts separate* (*so that the behavior of each is independent of the others' states*) *and whose parts then act so that they change towards forming connections of some type. Such a system is "self-organizing" in the sense, that it changes from parts separated to part joined*" [2]. The second meaning of self-organization concerns "*changing from a bad organization to good one*" [3]. "*The system would be "self-organizing", if a change were automatically made to the feedback, changing it from positive to negative; the whole would have changed from a bad organization to a good. Clearly, this type of "self-organization" is of peculiar to us. If we wish it to be a "good" one, we must first provide a criterion for distinguishing between the bad and the good, and then we must ensure, that the appropriate selection is made*" [2].

E-mail address: fgrab@prz.edu.pl.

0378-4371/$ – see front matter © 2010 Elsevier B.V. All rights reserved.
doi:10.1016/j.physa.2010.03.024

Table 1
Spatial–temporal context of self-organization of executive systems.

		Self-organization processes	
		Homogeneous	Heterogeneous
System resources	Unlimited	System without self-organization (the simple Malthus equation)	Special hypothetic but unrealistic cases
	Limited	Special case of self-organization (e.g. the logistic equation)	Arbitrary order self-organization (equation)

On the other hand, physical phenomena of self-organization in various executive complex systems are connected with specific spatial–temporal context. When considering an executive system, we often associate the concept of space with the system's resources whose performance in the ideal case is unlimited; yet, they are limited in the real environment. Time, however, is related to the processes occurring in the system. These processes are in the ideal case homogeneous and undisturbed, that is without self-organization effects. However, in reality they are heterogeneous and disturbed depending on the sensitivity of processes in the system to initial conditions. This spatial–temporal context of self-organization processes in systems leads to the following taxonomy, Table 1.

The Malthus equation and the classical logistic equation are only chain links of this taxonomy. The Malthus equation characterizes a simple system when the system's resources are unlimited but the processes are homogeneous, that is without self-organization effects. However, the original logistic equation concerns the rivalry of many homogeneous members of one species for limited resources of the system. In this case, self-organization of the members is homogeneous and can be described by a single characteristic performance of the system X, vs. the number of tasks N. It is known, however, that members (tasks) of one species can differ from one another. In reality they are usually heterogeneous, which results from people's different personalities, programming instruction, virus mutant, different packets in computer network, etc. When analyzing Table 1, we can notice lack of a general equation of arbitrary order which would describe the performance of a system with limited resources governed by heterogeneous self-organization processes. Therefore our proposal relates to a generalized formula of the logistic equation with limited external resources of the system and internal heterogeneous processes.

Natural systems are self-adaptive complex systems and unequalled archetype for designers of artificial systems. This means that they are equipped with mechanisms of both self-organization and self-optimization, which makes them autonomic. As a result of external and internal limitations, all systems self-organize, which usually leads to their degradation and departure from the state of equilibrium. Yet, the return of natural systems to the state of equilibrium ensures the mechanism of self-optimization. The situation is diametrically different in the case of artificial complex systems which are degraded as a results of self-organization but do not have inbuilt mechanisms of self-optimization. Therefore the work point of artificial system is far from equilibrium and the matching point between the performance of systems and the optimum number of realized tasks. Thus, it is a great challenge for contemporary science to equip artificial complex systems with mechanisms of self-optimization similar to those observed in natural systems. However, we should first understand the mechanisms of self-matching in a complex system. It is the aim of the second part of the paper.

This paper consists of five sections. The first one is a short introduction to the spatial–temporal context of physical models of systems, the second one presents the Malthusian model, its generalization and conversion from the macroscopic to microscopic from, whereas the third one deals with the generalized complex model based on the developed version of the logistic equation. The fourth section focuses on the analysis of the extended formula of the logistic equation and the fifth one provides conclusions.

2. From the macroscopic to microscopic form of the Malthus equation

The fundamental property of self-organizing systems is that the global character at macroscopic level emerges as a consequence of interactions between the system's elements at microscopic level. The spatial–temporal context of the physical model at microscopic level can be easily shown on the basis of queuing systems. In the general case of non-extensive complex systems, the service of each task can be characterized by microscopic parameters V_i, S_i, i and Z. V_i is the number of references (visits) to the ith component of the system during the service of each task, where in general $0 \leq i < \infty$, and $0 \leq V_i < \infty$, S_i is the time of serving the tasks by the system's ith component, where in the general case $0 < S_i < \infty$, and Z is the think time of a given process.

On the other hand, a simple system is a special case associated with unlimited resources and homogeneous processes determined by $i = \text{const}$, $S_i = \text{const}$, $V_i = \text{const}$ and $Z = \text{const}$. This means that a simple system can be characterized by the following thermodynamic equilibrium features: only permitted states can appear in the system, in the state of thermodynamic equilibrium all permitted states are equally probable, the number of possible states is finite, the entropy is extensive, and there are only short-term interactions between the states of the system. Therefore unlimited resource can be