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Edgeworth expansions of stochastic trading time

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Under most local and stochastic volatility models the underlying forward is assumed to be a positive function of a time-changed Brownian motion. It relates nicely the implied volatility smile to the so-called activity rate in the market. Following Young and DeWitt-Morette (1986) [8], we propose to apply the Duru–Kleinert process-cum-time transformation in path integral to formulate the transition density of the forward. The method leads to asymptotic expansions of the transition density around a Gaussian kernel corresponding to the average activity in the market conditional on the forward value. The approximation is numerically illustrated for pricing vanilla options under the CEV model and the popular normal SABR model. The asymptotics can also be used for Monte Carlo simulations or backward integration schemes.

1. Introduction

The central theme of this paper is the arbitrage free dynamics of forward prices and the numerical evaluation of simple financial securities. A forward contract is an agreement to pay a specified price at maturity T for a tradable asset X. If we define the discount factor as $D(t,T) = E^*\left[e^{-\int_t^T r(s)\mathrm{d}s}\right]$, with $\{r(s), 0 \le s \le T\}$ the spot short-term interest rate, then the forward price $F_T(t)$ for such a contract at time t $(0 \le t \le T)$ can be determined by

$$E^*\left[e^{-\int_t^T r(s)\mathrm{d}s}(X(T) - F_T(t))\right] = 0, \tag{1}$$

where $E^*[.]$ denotes the expectation with respect to the risk-neutral measure. Thus

$$X(t) - D(t,T)F_T(t) = 0, \quad \text{or} \quad F_T(t) = X(t)/D(t,T), \tag{2}$$

(see e.g. Ref. [1]). The time-evolution of the forward price $F_T(t)$ is partially determined by arbitrage free requirements. Indeed there exists a forward probability measure, equivalent to the risk-neutral measure, under which the forward price is a martingale. For a complete account on forward measure we refer to Ref. [2]. In the following, we omit in the notation of the forward price the dependence on T, and we will use $F(t)$ instead of $F_T(t)$, and f_0 will denote the forward price at time 0, $F(0)$.

Next to forward contracts, simple securities are traded in the markets on a regular basis. A plain vanilla European call option with pay-off $(x-K)_+$ e.g., gives the holder protection against high values of the underlying asset X at maturity date T. As the price of any asset divided by the discount factor $D(t,T)$ is a martingale under the corresponding forward measure

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(in financial terms, we say that the asset $D(t, T)$ is the numeraire); see e.g. Ref. [2], the price $V(f_0, K, T)$ of a European call option with strike K and expiry T at time 0 satisfies

$$\begin{aligned}\frac{V(f_0, K, T)}{D(0, T)} &= E_{f_0}\left[\frac{V(F(T), K, 0)}{D(T, T)}\right] \\ &= E_{f_0}\left[(F(T) - K)_+\right] \\ &= \int_K^{+\infty} p(f_0, f_T, T)(f_T - K)\mathrm{d}f_T,\end{aligned} \tag{3}$$

where E_{f_0} denotes the expectation with respect to the forward measure and where $p(f_0, f, T)$ is the transition density of the price process $\{F(t), t \geq 0\}$, or $p(f_0, f, T) = E_{f_0}[\delta(F(T) - f)]$.

The Black model assumes that the forward price of an asset follows the log-normal process

$$\mathrm{d}F(t) = \sigma_B F(t)\mathrm{d}W(t), \qquad F(0) = f_0 \tag{4}$$

where $\{W(t), t \geq 0\}$ is a Brownian motion in the forward measure; see e.g. Ref. [3] for more details.

From a practical perspective, the model has to be consistent with the prices of liquid (or vanilla) options quoted in the market. Although the Black model remains popular to manage books of vanilla options, the assumption of log-normal underlying price is in fact not appropriate. As a consequence, the quoted implied Black volatilities $\sigma_B(K, T)$ vary with the expiry and along the strikes (skew and smile). An implied Black volatility corresponds to the volatility parameter σ_B of the Black diffusion (4) that matches exactly the market price of a European option with strike K and maturity T; the term smile refers to U-shape of implied volatilities as a function of the strike while a skew exhibits monotone implied volatilities across strikes.

That is where local volatility models enter, as contrary to the Black model, they are able to incorporate a base skew. More general stochastic volatility models can be defined as

$$\mathrm{d}F(t) = v(t)\psi(F(t))\mathrm{d}W(t), \qquad F(0) = f_0, \qquad v(0) = v_0 \tag{5}$$

with ψ a continuously differentiable function, and with $\{v(t), t \geq 0\}$ another adapted process eventually correlated to F. Contrary to the Black model, they offer a good fit to the market smile and predict realistic dynamics of forward volatilities.

For most stochastic volatility models, a closed-form expression for the moment generating function (MGF) is available. This allows a fast calibration to the market smile, but at the cost of inverting the Fourier transform. Edgeworth expansions present a possible method to invert the Fourier transform analytically. It consists of truncating the cumulant representation of the MGF and expand the transition density $p(f_0, f_T, T)$ of the forward price F around the Gaussian kernel with the correct two first moments. In particular, this means that

$$\begin{aligned}p(f_0, f_T, T) &= E_{f_0}[\delta(F(T) - f_T)] \\ &= \int_{-\infty}^{+\infty} \mathrm{d}k E_{f_0}\left[\mathrm{e}^{-2\pi \mathrm{i}kF(T)}\right]\mathrm{e}^{2\pi \mathrm{i}kf_T} \\ &= \int_{-\infty}^{+\infty} \mathrm{d}k \exp\left(\sum_{j=1}^{+\infty} \frac{(-2\pi \mathrm{i}k)^j}{j!} C_j\right)\mathrm{e}^{2\pi \mathrm{i}kf_T},\end{aligned} \tag{6}$$

where C_j with $j \geq 1$ are the cumulants[1] of $F(T)$ and where we have used the Fourier representation of the Dirac delta function $\delta(x) = \int_{-\infty}^{+\infty} \mathrm{d}k\, \mathrm{e}^{-2\pi \mathrm{i}kx}$. Expanding the exponential function after the second cumulant, we obtain

$$\begin{aligned}p(f_0, f_T, T) &= \int_{-\infty}^{+\infty} \mathrm{d}k \exp\left(-2\pi \mathrm{i}kC_1 - 2\pi^2 k^2 C_2\right)\left(1 + \frac{8}{6}\pi^3 ik^3 C_3 + \cdots\right)\mathrm{e}^{2\pi \mathrm{i}kf_T} \\ &= \frac{1}{\sqrt{2\pi C_2}}\mathrm{e}^{-\frac{(f_T - C_1)^2}{2C_2}} - \frac{1}{6}C_3\frac{\partial^3}{\partial f_T^3}\left(\frac{1}{\sqrt{2\pi C_2}}\mathrm{e}^{-\frac{(f_T - C_1)^2}{2C_2}}\right) + \cdots.\end{aligned} \tag{7}$$

Unfortunately, this series converges slowly for heavy tailed distributions, which is the case for realistic stochastic volatility models, because the cumulants C_j with $j \geq 3$ are high compared to C_1 and C_2. If either the MGF is not available or a numerical inversion of the Fourier transform is not appropriate, asymptotic expressions for the implied Black volatility $\sigma_B(K, T)$ are suggested in the mathematical finance literature; see e.g. Refs. [4] or [5] among many others.

1 The first cumulant of a random variable X is precisely its expectation $E[X]$, the higher order cumulants coincide with the higher order centered moments $C_j = E\left[(X - E[X])^j\right]$ for $j > 1$. Note that the cumulants of higher order are related to the moments $M_j = E\left[X^j\right]$ by the polynomial relation

$$C_n = M_n - \sum_{i=1}^{n-1}\binom{n-1}{i-1} C_i M_{n-i}.$$