Physica A 389 (2010) 3336–3347

Contents lists available at ScienceDirect

Physica A

journal homepage: www.elsevier.com/locate/physa

Impact of initial lattice structures on networks generated by traces of random walks

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ARTICLE INFO

Article history:
Received 20 November 2009
Received in revised form 14 March 2010
Available online 31 March 2010

Keywords:
Network evolution
Random walks
Diffusion rate
Clustered structure

ABSTRACT

We show that the platform stage of network evolution plays a principal role in the topology of resulting networks generated by short-cuts stimulated by the movements of a random walker, the mechanism of which tends to produce power-law degree distributions. To examine the numerical results, we have developed a statistical method which relates the power-law exponent γ to random properties of the subgraph developed in the platform stage. As a result, we find that an important exponent in the network evolution is α, which characterizes the size of the subgraph in the form $V \sim t^{\alpha}$, where V and t denote the number of vertices in the subgraph and the time variable, respectively. 2D lattices can impose specific limitations on the walker's diffusion, which keeps the value of α within a moderate range and provides typical properties of complex networks. 1D and 3D cases correspond to different ends of the spectrum for α, with 2D cases in between. Especially for 2D square lattices, a discontinuous change of the network structure is observed, which varies according to whether γ is greater or less than 2. For 1D cases, we show that emergence of nearly complete subgraphs is guaranteed by $\alpha < 1/2$, although the transient power-law is permitted at low increase rates of edges. Additionally, the model exhibits a spontaneous emergence of highly clustered structures regardless of its initial structure.

1. Introduction

Over the last few years, empirical studies of large-scale networks, such as the Internet, biological systems and social networks, have noted common topological properties, for example, a power-law degree distribution (the scale-free property) and small-world phenomena [1–3]. These findings have stimulated extensive studies on networks in the real world, because the network topology plays a significant role in processes taking place on large-scale systems in such diverse research areas as the robustness of a network with respect to attacks or failures [4–6], epidemic processes [7–10], and phase transitions in opinion dynamics [11–13].

Mathematical network models are effective tools to study how such a universal structure emerges. Continuous addition of new vertices to networks, i.e., the growth property, has played a large role in the development of mathematical network models [14], as not only can the growth property describe the ordinal growth features of networks, it can also explain the emergence of scale-free properties. However, the growth property by itself cannot explain all aspects of real networks, for there are other possible properties of real networks, such as exhibiting a highly clustered structure, e.g., two nodes having a common neighbor are likely to be joined, and the existence of positive or negative correlations between nodes [15,16]. Moreover, power-law exponents can assume different values corresponding to specific conditions of the network. To model diverse aspects of networks, it is natural to take into account local properties of networks. For example, a highly

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doi:10.1016/j.physa.2010.03.027

clustered structure can be modeled by making various assumptions, for example, that the original spatial structure is highly clustered [17], two nodes that have a common neighbor are likely to be connected [18–22], networks exhibit a hierarchical organization [23], and only activated nodes can connect with newly attached nodes in the growth process [24]. It is also natural to expect that local events, such as the creation of short cuts between pre-existing vertices, and eliminations of edges and vertices, are indispensable for studies of time-dependent dynamics in networks.

This paper studies a simple network model where random transits, represented by a random walker, stimulate short-cut creations in the network [25,26]. The rule is that new edges are created between the vertex where the random walker is situated currently and the vertex where the random walker was two time-steps before. (A detailed description will be presented in the next section.) This model considers the following two hypothetical features of real networks. (1) The frequency of transit between elements of the system determines the rise and fall of the local connectivity of the system. For example, vertices that are already of high degree are likely to acquire newer edges, owing to their high ability to attract a random walker along one of their existing edges. (2) Vertices geographically close to each other are tightly connected. In order to model this feature, lattice points prepared as the initial structure are assumed to be always connected, regardless of the movements of the random walker. Geographical constraint is probably significant for networks [27–29] where the interaction between elements is via physical objects (e.g., technological and neural networks); on the other hand, there may be networks that do not care about the physical location of vertices (e.g., the World Wide Web). The present paper provides one example where the platform stage of network evolution affects the resulting network structure, although only idealized situations are considered, such as assuming a regular lattice structure, rather than the actual situation. The investigated networks are able to make the transition from a regular lattice to a more complex network in which a spatial structure is embedded.

In a previous paper, it was shown that network properties of a 1D lattice, such as degree distribution, vertex–vertex distances, and the migration process of sub-networks caused by the extinction of edges, are different from those of 2D square lattices [26]. However, the question of what determines such differences has not been answered. To answer that question, in this paper, focus is maintained on the impact of the initial lattice structure on network evolution by avoiding complicated situations caused by the process of eliminating old edges, which was considered in the preceding study. Instead of considering the elimination of edges, an ''edge creation probability'' that controls the creation of edges per step is introduced, whereas, in earlier work, the random movement of the walker definitely added edges to the graph at each step. The introduction of the ''edge creation probability'' to some extent frees the movement of the walker from the influence of its last movement.

This paper is organized as follows. In the next section, the model is defined and an overview of the resulting networks is given. Investigations in this paper are carried out for networks formed from a 1D lattice, a 2D square lattice, a 2D triangular lattice, and a simple cubic lattice, which are reviewed in Section 2. In Section 3, some exponents, α, β and z, that describe the growth rates of networks created by the walker are defined, and general relations, which are not always specific to the model, between the exponents and some restrictions on them, are provided. In Section 4, numerical results for degree distributions are examined using the relations introduced in Section 3. Here, a condition for a constant rate of increase in the number of edges is provided, discontinuous changes in 2D square lattices corresponding to whether γ is larger or smaller than 2 are explained, and the emergence of nearly complete subgraphs in 1D lattices is considered. In Section 5, calculation results of the number of vertices within a certain distance from a vertex of maximum degree are given, and limitations on the maximum vertex degree are discussed. In Section 6, a relation is derived between clustering coefficients and the increase rate of edges, and we compare the relation with numerical results. Section 7 summarizes these results.

2. Model

The model is defined by the transformation rule of a graph G_t, that is determined by the movements of a random walker on the graph G_t, where t denotes a discrete time t $(t = 0, 1, 2, \ldots)$. The transformation rule is:

1. At initial time $t = 0$, an initial lattice with no boundary is given as G_0. This paper deals with four types of G_0: a 1D lattice, a 2D square lattice, a 2D triangular lattice, and a simple cubic lattice. A random walker is assumed to be located at one particular vertex in G_0 at time $t = 0$.
2. At each time step t, the random walker moves with equal probability to an adjacent vertex in G_t. Let the walker's location at time t be $\mathbf{x}_t$. Then the movement $\mathbf{x}_t \to \mathbf{x}_{t+1}$ results in the probabilistic creation of new edges as follows. If there is no edge between $\mathbf{x}_{t-1}$ and $\mathbf{x}_{t+1}$, a new edge is created with probability p_e between $\mathbf{x}_{t-1}$ and $\mathbf{x}_{t+1}$, which results in a revised graph G_{t+1} from G_t. If there already exists an edge between $\mathbf{x}_{t-1}$ and $\mathbf{x}_{t+1}$, G_{t+1} remains the same as G_t (See Fig. 1).

Iteration of the second process results in a monotonic increase in the number of edges with time. Note that the random walker is supposed to move in G_t which includes G_0. This rule gives vertices of large degree an advantage in attracting the random walker by their edges and thereby to gain new edges, whereas a vertex cannot gain edges at all without a visit from the random walker. Consequently, the region in which each vertex has edges created by the walker's passing corresponds to the region that the random walker has visited at least once. The graph to be investigated in the following sections is such an evolving subgraph g_t in G_t, that consists of vertices with edges created by the random walker's passing. Typical examples of g_t created by this rule are illustrated in Figs. 1(b), 2 and 3, each of which eventuated from a 2D triangular lattice, a simple cubic lattice, and a 1D lattice, respectively.