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Scattering by one-dimensional smooth potentials: between WKB and Born approximation

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Abstract

The paper discusses the applicability of WKB and Born (small perturbations) approximations in the problem of the backscattering of quantum particles and classical waves by one-dimensional smooth potentials with small amplitudes compared to the energy of the incident particle (above-barrier scattering). Both deterministic and random potentials are considered. The dependence of the reflection coefficient and localization length on the amplitude and the longitudinal scale of the scattering potential is investigated. It is shown that perturbation and WKB theories are inconsistent in the above-barrier backscattering problem. Not only the solutions but the regions of validity of both methods as well depend strongly on the details of the potential profile, and are individual for each potential. For deterministic potentials, a simple criterion that allows determining the boundary between the applicability domains of WKB and Born approximations is found.

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1. Introduction

Two approximate methods are most often used in solving quantum mechanical and electrodynamical scattering problems. First one, the perturbation theory [1], is valid when the amplitude, V_0, of the scattering potential is small as compared to the energy, E, of the particle, so that a solution is sought as a series in powers of the small parameter

$$\delta = \frac{V_0}{E} \ll 1. \tag{1}$$

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The first term of this series is known as Born approximation.

The second method is WKB (quasiclassical) approximation [1–4], which is applied when the potential is a smooth function of coordinates, i.e. when the characteristic longitudinal scale of its variation, L, is large as compared to the characteristic wavelength $2\pi k_0^{-1} = 2\pi E^{-1/2}$, and the solution can be presented as an asymptotic expansion in powers of the small parameter

$$\varepsilon = \frac{1}{k_0 L} \ll 1. \tag{2}$$

It may appear from Eqs. (1) and (2) that the perturbation theory does not impose any restrictions on the dimensionless inverse scale ε, while WKB method is applicable for any small values of the dimensionless amplitude δ and, therefore, when simultaneously $\delta \ll 1$ and $\varepsilon \ll 1$, both approximations should be valid and give the same result. It is known, however [1,5–7], that if the longitudinal scale of the potential variations increases ($\varepsilon \to 0$) the convergence condition of the perturbation theory series is violated, no matter how small the fixed amplitude δ is. On the other hand, if parameter ε is fixed ($\varepsilon \ll 1$) and the amplitude tends to zero ($\delta \to 0$), the WKB approximation breaks down [1,5–7]. (Note that in papers [8–11] attempts have been made to construct the approximate theory including both Born and WKB theories. As shown in Refs. [5–7] they are actually incorrect in the WKB region.) This brings up the question: What are the regions of validity of WKB and small perturbation approximations when the scattering on a smooth ($\varepsilon \ll 1$) potential with small amplitude ($\delta \ll 1$) is concerned?

In the present paper we discuss the applicability of WKB and Born approximations for one-dimensional above-barrier scattering problems with different types of potentials, both deterministic and random, in the ballistic and localized regimes. It is shown that in the quasiclassical region, Eq. (2), the reflection coefficient, R, is extremely sensitive to the exact shape of the potential profile. When simultaneously $\varepsilon \ll 1$ and $\delta \ll 1$ there is no universal characteristic dependence $R(\delta, \varepsilon)$, and this function is quite individual for each given potential. This is in contrast to the case of tunnelling ($\delta > 1$), when WKB approximation is robust in the sense that the transmission and reflection coefficients are determined by the characteristic height and width of the barrier and practically independent of the details of its shape. The surprising thing is that in the case of the above-barrier scattering even the regions of validity of WKB theory and Born approximation are essentially different for different potentials, so that universal inequalities restricting the applicability of the methods do not exist.

2. Basic equations

We consider the stationary one-dimensional Schrödinger equation:

$$\frac{d^2\psi}{dx^2} + [E - V(x)]\psi = 0, \tag{3}$$

where $E > 0$ is the energy (units $\hbar = 2m = 1$ are used), potential $V(x)$ is an analytical bounded function with a characteristic amplitude $V_0 = |V(x)|_{\max}$ and a single characteristic longitudinal scale L ($L^{-1} \sim |V'|/V_0$). By introducing variable $z = k_0 x$ and function $U(x/L) = U(\varepsilon z) \equiv V(x)/V_0$, we reduce Eq. (3) to the dimensionless form

$$\psi'' + [1 - \delta U(\varepsilon z)]\psi = 0, \tag{4}$$

where primes stand for derivatives with respect to z. Clearly, $U(\varepsilon z)$ is of the order of unity, while its derivative with respect to z is of the order of ε. In what follows we consider the above-barrier scattering in the sense that $\delta < 1$ (more precisely, the inequality $\delta\varepsilon/(1-\delta) \ll 1$ is necessary for WKB approximation to be valid for all real z [5]).

In the Born approximation, the reflection coefficient for a quantum particle (or classical wave) is given by [1]

$$R_{\mathrm{Born}} = \frac{\delta^2}{4} \left| \int_{-\infty}^{\infty} U(\varepsilon z) e^{2iz} \, dz \right|^2. \tag{5}$$

Eq. (5) represents the first term of the perturbation series. When the amplitude of $2k_0$-harmonic (corresponding to the first resonant Bragg backscattering) in the power spectrum of the potential