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Hurst exponents, Markov processes, and fractional Brownian motion

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Abstract

There is much confusion in the literature over Hurst exponents. Recently, we took a step in the direction of eliminating some of the confusion. One purpose of this paper is to illustrate the difference between fractional Brownian motion (fBm) on the one hand and Gaussian Markov processes where $H \neq \frac{1}{2}$ on the other. The difference lies in the increments, which are stationary and correlated in one case and nonstationary and uncorrelated in the other. The two- and one-point densities of fBm are constructed explicitly. The two-point density does not scale. The one-point density for a semi-infinite time interval is identical to that for a scaling Gaussian Markov process with $H \neq \frac{1}{2}$ over a finite time interval. We conclude that both Hurst exponents and one-point densities are inadequate for deducing the underlying dynamics from empirical data. We apply these conclusions in the end to make a focused statement about 'nonlinear diffusion'.

Keywords: Markov processes; Fractional Brownian motion; Scaling; Hurst exponents; Stationary and nonstationary increments; Autocorrelations

1. Introduction

The necessity of stationary increments for fractional Brownian motion (fBm) has been emphasized in books [1] and papers [2] by mathematicians, but also is sometimes not stated [3]. Books [4] and papers [5] by physicists tend to ignore the question altogether and to assume, without justification and incorrectly, that $H \neq \frac{1}{2}$ always implies long-time correlations. In this paper, we emphasize that the essential point in long-time correlations is stationarity of the increments, not scaling. Scaling makes life simpler, when it occurs, but is irrelevant for correlations: classes of stochastic dynamical systems with long-time correlations exist without scaling, while processes with no memory at all, Markov processes, can scale perfectly with a Hurst exponent $H \neq \frac{1}{2}$ [6]. Our point is that evidence for scaling, taken alone, tells us nothing whatsoever about the existence of long-time correlations. The basic question to be answered first, in both data analysis and theory, is: are the

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increments stationary or nonstationary? To achieve both unity and clarity, we begin with the mathematicians' usual definition of scaling of a stochastic process and then show how it leads naturally to the physicists' definition.

2. Hurst exponent scaling

We define scaling (self-similar processes) starting from the mathematicians' standpoint [1] and show that it is equivalent to our definition [6] in terms of densities.

A stochastic process $x(t)$ is said to scale with Hurst exponent H if [1]

$$x(t) = t^H x(1), \tag{1}$$

where by equality we mean equality 'in distribution'. We next define what that means in practice.

The one-point distribution $P(x,t)$ reflects the statistics collected from many different runs of the time evolution of $x(t)$ from a specified initial condition $x(t_0)$, but does not describe correlations or lack of same. The (one-point) density $f_1(x,t)$ of the distribution is defined by $f_1(x,t) = \mathrm{d}P/\mathrm{d}x$. Given any dynamical variable $A(x,t)$, averages of A are calculated via

$$\langle A(t)\rangle = \int_{-\infty}^{\infty} A(x,t) f_1(x,t)\,\mathrm{d}x. \tag{2}$$

We restrict to x-independent drift coefficients. Let $R(t)$ denote the drift coefficient in $x(t)$. Since the drift in

$$\langle x(t)\rangle = x(t_0) + \int_{t_0}^{t} R(s)\,\mathrm{d}s \tag{3}$$

depends only on t, then it is trivial to remove it from the stochastic process by choosing instead of $x(t)$ the Martingale variable $z(t) = x(t) - \int R(s)\,\mathrm{d}s$. In what follows we will write '$x(t)$' with the assumption that the drift has been removed, $\langle x(t)\rangle = x(t_0)$. We will also take $x(t_0) = 0$, so that more generally our $x(t)$ must be interpreted as $x(t) - x(t_0)$ if $x(t_0) \neq 0$. i.e., we are generally using $x(t)$ to mean the random variable $z(t) = x(t) - x(t_0) - \int \langle R\,\mathrm{d}t\rangle$.

From (1), the moments of x must obey

$$\langle x^n(t)\rangle = t^{nH}\langle x^n(1)\rangle = c_n t^{nH}. \tag{4}$$

Combining this with

$$\langle x^n(t)\rangle = \int x^n f_1(x,t)\,\mathrm{d}x \tag{5}$$

we obtain [6]

$$f_1(x,t) = t^{-H} F(u), \tag{6}$$

where the scaling variable is $u = x/t^H$. In particular, with no average drift, so that we can choose $\langle x(t)\rangle = x(t_0) = 0$, the variance is simply

$$\sigma^2 = \langle x^2(t)\rangle = \langle x^2(1)\rangle t^{2H}. \tag{7}$$

This explains what is meant by Hurst exponent scaling, and also specifies what is meant that (1) holds 'in distribution'. In all that follows, equations in random variables $x(t)$ like (1), solutions of stochastic differential equations (sdes), and increment equations as we shall write down in part 3 below, are all to be understood as holding 'in distribution'.

We ignore Levy distributions here because they are not indicated by our recent financial data analysis [7]. That analysis suggests diffusive processes, Markov processes. For discussions of Levy distributions, see Scalas et al. [8,9].

Empirically, the best evidence for scaling would be a data collapse of the form $F(u) = t^H f_1(x,t)$. Next best but weaker is to look for scaling in a finite number of the moments $\langle x^n(t)\rangle$. It is important to understand that Hurst exponent scaling, taken alone, tells us nothing about the underlying stochastic dynamics. In particular,