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Path integral approach to Asian options in the Black–Scholes model

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ABSTRACT

We derive a closed-form solution for the price of an average strike as well as an average price geometric Asian option, by making use of the path integral formulation. Our results are compared to a numerical Monte Carlo simulation. We also develop a pricing formula for an Asian option with a barrier on a control process, combining the method of images with a partitioning of the set of paths according to the average along the path. This formula is exact when the correlation is zero, and is approximate when the correlation increases.

1. Introduction

Since the beginning of financial science, stock prices, option prices and other quantities have been described by stochastic and partial differential equations. Since the 1980s however, the path integral approach has been introduced to the field. The path integral formalism was created by Richard Feynman [1] in quantum physics. Earlier, Norbert Wiener [2], in his studies on Brownian motion, used a type of functional integral, that turns out to be a special case of the Feynman path integral (see also Mark Kac [3], and for a general overview see Kleinert [4] and Schulman [5]). Path integrals were introduced in finance by Jan Dash, who developed applications related to the Black–Scholes model [6] and the one-factor term-structure-constrained model [7]. This fundamental work led to an increased interest in path integrals in finance [8–14]. One of the most challenging types of derivatives are the so-called path-dependent options, which were first studied by Linetsky [15]. Important classes of path-dependent options are Asian and barrier options.

Asian options are exotic options for which the payoff depends on the average price of the underlying asset during the lifetime of the option [16,11,17,18]. One distinguishes between *average price* and *average strike* Asian options. The payoff of an average price is given by $\max(\bar{S}_T - K, 0)$ and $\max(K - \bar{S}_T, 0)$ for a call and put option respectively. Here K is the strike price and $\bar{S}_T$ denotes the average price of the underlying asset at maturity T. $\bar{S}_T$ can either be the arithmetical or geometrical average of the asset price. Average price Asian options cost less than plain vanilla options. They are useful in protecting the owner from sudden short-lasting price changes in the market for example due to order imbalances [19]. Average strike options are characterized by the following payoffs: $\max(S_T - \bar{S}_T, 0)$ and $\max(\bar{S}_T - S_T, 0)$ for a call and put option respectively, where S_T is the price of the underlying asset at maturity T. Barrier options are options with an extra boundary condition. If the asset price of such an option reaches the barrier during the lifetime of the option, the option becomes worthless, otherwise the option has the same payoff as the option on which the barrier has been imposed (for more information on exit-time problems see Ref. [20] and the references therein).

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In Section 2 we treat the geometrically averaged Asian option. In Section 2.1 the asset price propagator for this standard Asian option is derived within the path integral framework in a similar fashion as Linetsky [15] derives it for the weighted Asian option. In Section 2.2 we present an alternative derivation of this propagator using a stochastic calculus approach. This propagator now allows us to price both the average price and average strike Asian option. For both types of options this results in a pricing formula which is of the same form as the Black–Scholes formula for the plain vanilla option. Our result for the option price of an average price Asian option confirms the result obtained by Linetsky [15] and Lipton [21]. For the average strike option no formula of this simplicity exists as far as we know. Our derivation and analysis of this formula is presented in Section 2.3, where our result is checked with a Monte Carlo simulation. In Section 3 we impose a boundary condition on the Asian option in the form of a barrier on a control process, and check whether the formalism introduced by Linetsky is still valid when the boundary condition is imposed on the propagator derived in Section 2, using the method of images. Finally in Section 4 we draw conclusions.

2. Geometric Asian options in the Black–Scholes model

2.1. Partitioning the set of all paths

The path integral propagator is used in financial science to track the probability distribution of the logreturn $x_t = \log(S_t/S_0)$ at time t, where S_0 is the initial value of the underlying asset. This propagator is calculated as a weighted sum over all paths from the initial value $x_0 = 0$ at time $t = 0$ to a final value $x_T = \log(S_T/S_0)$ at time $t = T$:

$$\mathcal{K}\left(x_T, T|0, 0\right) = \int \mathcal{D}x \, \exp\left(-\int_0^T \mathcal{L}_{BS}\left[x(t)\right] \, \mathrm{d}t\right). \tag{1}$$

The weight of a path, in the Black–Scholes model, is determined by the Lagrangian

$$\mathcal{L}_{BS}\left[x(t)\right] = \frac{1}{2\sigma^2}\left[\dot{x} - \left(\mu - \frac{\sigma^2}{2}\right)\right]^2. \tag{2}$$

where μ is the drift and σ is the volatility appearing in the Wiener process for the logreturn [8].

For Asian options, the payoff is a function of the average value of the asset. Therefore we introduce $\bar{x}_T = \log\left(\bar{S}_T/S_0\right)$ as the logreturn corresponding to the average asset price at maturity T. When $\bar{S}_T$ is the geometric average of the asset price, then $\bar{x}_T$ is an algebraic average.

$$\bar{x}_T = \frac{1}{T}\int_0^T x(t)\mathrm{d}t. \tag{3}$$

The key step to treat Asian options within the path integral framework is to partition the set of all paths into subsets of paths, where each path in a given subset has the same average $\bar{x}_T$. Summing over only these paths that have a given average $\bar{x}_T$ defines the conditional propagator $\mathcal{K}\left(x_T, T\ |0, 0|\, \bar{x}_T\right)$. This technique was developed in Ref. [22], it was developed for the purpose of reducing many complicated path integrals to simple ordinary integrals, and has led to the powerful calculus of variational perturbation theory [4]. Later Linetsky [15] observed that this technique could be used in the context of Asian options. $\mathcal{K}\left(x_T, T\ |0, 0|\, \bar{x}_T\right)$ becomes:

$$\mathcal{K}\left(x_T, T\ |0, 0|\, \bar{x}_T\right) = \int \mathcal{D}x\, \delta\left(\bar{x}_T - \frac{1}{T}\int_0^T x(t)\, \mathrm{d}t\right) \exp\left(-\int_0^T \mathcal{L}_{BS}\left[x(t)\right]\, \mathrm{d}t\right). \tag{4}$$

This is indeed a partitioning of the sum over all paths:

$$\mathcal{K}\left(x_T, T|0, 0\right) = \int_{-\infty}^{\infty} \mathrm{d}\bar{x}_T\ \mathcal{K}\left(x_T, T\ |0, 0|\, \bar{x}_T\right). \tag{5}$$

The delta function in the sum $\int \mathcal{D}x$ over all paths picks out precisely all the paths that will have the same payoff for an Asian option.

The calculation of $\mathcal{K}\left(x_T, T\ |0, 0|\, \bar{x}_T\right)$ is straightforward; when the delta function is rewritten as an exponential,

$$\mathcal{K}\left(x_T, T\ |0, 0|\, \bar{x}_T\right) = \int_{-\infty}^{\infty} \frac{\mathrm{d}k}{2\pi} \mathrm{e}^{\mathrm{i}kx} \int \mathcal{D}x\, \exp\left(-\int_0^T \left(\mathcal{L}_{BS}\left[x(t)\right] + \frac{1}{T}\mathrm{i}kx(t)\right) \mathrm{d}t\right), \tag{6}$$

the resulting Lagrangian is that of a free particle in a constant force field in 1D. The resulting integration over paths is found by standard procedures [23]:

$$\mathcal{K}\left(x_T, T\ |0, 0|\, \bar{x}_T\right) = \frac{\sqrt{3}}{\pi\sigma^2 T} \exp\left\{-\frac{1}{2\sigma^2 T}\left[x_T - \left(\mu - \frac{\sigma^2}{2}\right)T\right]^2 - \frac{6}{\sigma^2 T}\left(\bar{x}_T - \frac{x_T}{2}\right)^2\right\}, \tag{7}$$

and corresponds to the result found by Kleinert [4] and by Linetsky [15].