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Wavelet entropy and fractional Brownian motion time series

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Abstract

We study the functional link between the Hurst parameter and the normalized total wavelet entropy when analyzing fractional Brownian motion (fBm) time series—these series are synthetically generated. Both quantifiers are mainly used to identify fractional Brownian motion processes [L. Zunino, D.G. Pérez, M. Garavaglia, O.A. Rosso, Characterization of laser propagation through turbulent media by quantifiers based on the wavelet transform, Fractals 12(2) (2004) 223–233]. The aim of this work is to understand the differences in the information obtained from them, if any.

Keywords: Fractional Brownian motion; Wavelet theory; Hurst parameter; Mean normalized total wavelet entropy

1. Introduction

When studying the laser beam propagation through a laboratory-generated turbulence [1] we have introduced two quantifiers: the *Hurst parameter*, H, and the *normalized total wavelet entropy* (NTWS), S_{WT}. The former quantifier was introduced to test how good the family of *fractional Brownian motion* [2] (fBm) processes models the wandering of such laser beam, while the NTWS is a more general quantifier aimed to study any given dynamic system [3]. Also, in a recent work we have analyzed the dynamic case: the laboratory-generated turbulence was set up to change in time [4]. We have observed that these quantifiers are correlated, but at the time only a qualitative argument was given. Furthermore, each one of these quantifiers have been used separately to obtain information from biospeckle phenomenon [5,6].

The fBm is the only one family of processes which is self-similar, with stationary increments, and Gaussian [7]. The normalized family of these Gaussian processes, B^H, is the one with $B^H(0)=0$ almost surely, $\mathbb{E}[B^H(t)]=0$, and covariance

$$\mathbb{E}[B^H(t)B^H(s)] = \tfrac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H}), \tag{1}$$

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for $s, t \in \mathbb{R}$. Here $\mathbb{E}[\cdot]$ refers to the average with Gaussian probability density. The power exponent H has a bounded range between 0 and 1. These processes exhibit memory, as can be observed from Eq. (1), for any Hurst parameter but $H = \frac{1}{2}$. In this case successive Brownian motion increments are as likely to have the same sign as the opposite, and thus there is no correlation. Otherwise, it is the Brownian motion that splits the family of fBm processes in two. When $H > \frac{1}{2}$ the correlations of successive increments decay hyperbolically, and their sum diverges, this sub-family of processes have long-memory. Besides, consecutive increments tend to have the same sign, these processes are *persistent*. For $H < \frac{1}{2}$, the correlations of the increments also decay hyperbolically but their sum is summable, and this sub-family presents short-memory. Since consecutive increments are more likely to have opposite signs, it is said that these are *anti-persistent*.

The wavelet analysis is one of the most useful tools when dealing with data samples. Thus, any signal can be decomposed by using a diadic discrete family $\{2^{j/2}\psi(2^j t - k)\}$—an *orthonormal* basis for $L^2(\mathbb{R})$—of translations and scaling functions based on a function ψ: the mother wavelet. This wavelet expansion has associated wavelet coefficients given by $C_j(k) = \langle \mathscr{S}, 2^{j/2}\psi(2^j \cdot -k)\rangle$. Each resolution level j has an associated energy $\mathscr{E}_j = \sum_k |C_j(k)|^2$. The *relative wavelet energy*, RWE, is

$$p_j = \frac{\mathscr{E}_j}{\mathscr{E}_{\text{tot}}}, \tag{2}$$

with $j \in \{-N, \ldots, -1\}$, where $N = \log_2 M$ with M the number of sample points, and $\mathscr{E}_{\text{tot}} = \sum_{j=-N}^{-1} \mathscr{E}_j$ is the total energy. Thus the NTWS is defined as (see Ref. [1] and references therein)

$$S_{\text{WT}} = -\sum_{j=-N}^{-1} p_j \cdot \log_2 p_j / S^{\max} \quad \text{with} \quad S^{\max} = \log_2 N. \tag{3}$$

For a signal originated from a fBm the energy per resolution level j and sampled time k can be calculated using the formalism introduced in Ref. [8], see Appendix,

$$\mathbb{E}|C_j(k)|^2 = 2\Gamma(2H+1)\sin(\pi H)2^{-j(2H+1)} \int_0^\infty \frac{|\hat{\psi}|^2(v)}{v^{2H+1}}, \tag{4}$$

for any mother wavelet election satisfying $\int_{\mathbb{R}} \psi = 0$. As you can observe the latter expression is independent of k. That is true for all stochastic processes with stationary increments [9,10]. On the other hand, a similar power-law behavior, $\mathbb{E}|C_j(k)|^2 \propto 2^{-j\alpha}$, is verified for stationary process displaying long-range dependence (LRD).[1] So, our results can also be applied for fractional Gaussian noises (fGn), for example.

From (4) the RWE for a finite data sample is

$$p_j = 2^{(j+1)(1+2H)} \frac{1 - 2^{-(1+2H)}}{1 - 2^{-N(1+2H)}}. \tag{5}$$

Observe that these coefficients are independent on wavelet basis. It can be easily computed from definition (2) and Eq. (4)—see again Appendix. And so it does the NTWS,

$$\begin{aligned} S_{\text{WT}}(N, H) = {} & \frac{1}{\log_2 N}(1+2H)\left[\frac{1}{2^{1+2H}-1} - \frac{N}{2^{N(1+2H)}-1}\right] \\ & - \frac{1}{\log_2 N}\log_2\left[\frac{1-2^{-(1+2H)}}{1-2^{-N(1+2H)}}\right]. \end{aligned} \tag{6}$$

As it was expected the entropy decreases when H increases, with H measuring the level of order of the signal.

2. Simulations and tests

To test the functional relation between the Hurst exponent and NTWS we have simulated 50 fBm data samples [11] for each $H \in \{0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9\}$. Since we have examined data of 5000 points in length in Ref. [1], these samples are set to the same length. For each set we estimate H and S_{WT}. Moreover, we

[1] Remember that $\alpha = 2H + 1$ for self-similar processes with stationary increments and $\alpha = 2H - 1$ for long-range dependent processes.