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Unitarily inequivalent vacua in Bose–Einstein condensation of trapped gases

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Abstract

We approach the problem of the Bose–Einstein condensation of neutral atoms in an external trap, which is a finite system in size, from the viewpoint of quantum field theory. The phenomenon is considered as a spontaneous symmetry breakdown. The main result of this paper is that the vacua for this system with finite size and finite number of trapped neutral atoms, parametrized by different phases of the order parameter, are orthogonal to each other, namely the Fock spaces associated with each vacuum are unitarily inequivalent to each other. In the proof, the zero-energy mode which is the Nambu–Goldstone mode, and is discrete due to the presence of the trap, plays a crucial role. We introduce the discrete zero-energy mode in the formulation of the generalized Bogoliubov transformation and introduce an additional infinitesimal symmetry-breaking term in the Hamiltonian to control the zero-energy mode properly.

Keywords: Bose–Einstein condensation; Unitarily inequivalent vacua; Spontaneous symmetry breakdown

1. Introduction

Bose–Einstein condensates (BECs) have been realized in trapped neutral atomic gas systems [1–3]. Many static and dynamical properties of BECs have been investigated theoretically [4]. These systems give us an ideal test field for quantum many-body theory, e.g., quantum statistical mechanics, quantum field theory (QFT), thermal field theory, and so on. A gas system which is dilute and in which the interaction between atoms is weak exhibits a distinct contrast to any other condensed system, ever observed before the BEC was achieved, including the superfluid of ^{4}He. This weakly interacting system allows us to use a perturbative approach, and it is of great advantage for theoretical investigations of condensate phenomena.

The BEC is described as a spontaneous breakdown of a global phase symmetry in the formulation of QFT [5]. Usually phenomena with spontaneous breakdown of symmetry in QFT are formulated for spatially homogeneous systems of infinite size. Recall the formulation of QFT for homogeneous systems [6,7]. Suppose that there is a complex field ψ and that the Hamiltonian (or the action) is invariant under its global phase

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transformation. Then for some classes of interaction models, we find two phases, i.e., the Wigner phase (symmetric one above the critical temperature) and the Goldstone one (broken one below the critical temperature). These phases are characterized by the vacuum expectation value of ψ, denoted by $\langle\psi\rangle$, which corresponds to the order parameter. In the Goldstone phase, we have $\langle\psi\rangle = e^{i\theta}v$ (v, θ: real constants), while $\langle\psi\rangle$ disappears in the Wigner phase. There are degenerate vacua for the Goldstone phase, each vacuum being labeled by the continuous parameter θ. The Goldstone phase is analogous to the ferromagnetic one of the spin system in which the angle indicating the direction of magnetization is a continuous parameter, while the Wigner phase corresponds to the paramagnetic phase.

A key concept of the QFT formulation is the existence of unitarily inequivalent vacua. Let us denote the vacuum of the Wigner phase by $|0\rangle$, and the vacua of the Goldstone phase by $|\Omega(\theta)\rangle$ for each θ. Then it can be shown that the inner products among $|\Omega(\theta)\rangle$ with different θ's and $|0\rangle$ such as $\langle\Omega(\theta)|\Omega(\theta')\rangle$ and $\langle 0|\Omega(\theta)\rangle$ are proportional to $\exp[-\gamma\delta(\vec{k}=\vec{0})]$ [6] . Here γ is some positive number, and $\delta(\vec{k}=\vec{0})$ is the delta function in momentum space at $\vec{k}=\vec{0}$ and is therefore equal to $(2\pi)^3 V$ where V is the volume. Thus, all the vacua are orthogonal to each other in the infinite volume limit $V \to \infty$. Furthermore, one can prove that any Fock state built on one vacuum is orthogonal to all Fock states built on the other vacua, and therefore cannot be a superposition of the Fock states built on these different vacua. This is the description of how a spontaneous breakdown of symmetry for a homogeneous and an infinite size system is treated in QFT and how the concept of unitarily inequivalent vacua appears [8].

We turn to the problem of the trapped BEC. Then the system is neither homogeneous nor infinite in size due to the trapping potential. One may suspect that for the trapped BEC the usual QFT formulation and concepts of spontaneous symmetry breakdown in the preceding paragraphs do not work and that there is no need for unitarily inequivalent vacua. We remark that the ''system size'' which is the spatial extension of the system should not be confused with the ''volume'' in this paper.

It is noteworthy in this respect that an interference between two trapped condensates has been observed [9,10], and that two or more condensates with different phases also interfere [11–13]. From the viewpoint of QFT, the experiments indicate that the order parameter $\langle\psi\rangle$ has a definite phase. This fact suggests that there exist unitarily inequivalent vacua for the trapped BEC, guaranteeing a definite phase for each vacuum.

When a continuous symmetry is broken spontaneously, the Nambu–Goldstone (NG) mode appears, which is required by the Goldstone theorem [14], irrespective of whether the system size is infinite or finite. We have shown that for the trapped BEC in which all the modes are discrete the discrete zero mode cannot be suppressed and has to be introduced explicitly to satisfy the canonical commutation relations (CCRs) [5]. Two approaches to treat the zero mode are known to exist: one is called the generalized Bogoliubov approach using the zero-energy mode (ZEM) [5], and the other is the Bogoliubov-de Gennes one in which the zero mode is represented by a pair of the quantum coordinates [15,16]. The relationship between the two approaches is revealed in Ref. [17]. The ZEM plays the role of the NG mode, which is evident from the argument that the Ward–Takahashi (WT) relation [6,7,18] holds if the propagators include the ZEM properly [5,19]. But it is known that the divergence originating from the ZEM appears in the propagators. We expect that this divergence can be removed by the renormalization procedure, and have demonstrated the renormalization in the calculation of the tadpole diagram [5].

The purpose of this paper is to show that the vacua yielding different phases of the order parameter are orthogonal to each other for the trapped BEC, using the ZEM. Namely, the concept of unitarily inequivalent vacua is not inherent to the infinity in size, but is possible also for spatially finite-size systems. The mechanism for this arises from the collective contribution of the ZEM. If we omit the ZEM from our formulation, which is inconsistent because the CCRs as the very fundamental principle of QFT are violated, the vacua would not be orthogonal to each other. It can also be noted that the vacuum orthogonality is true even for a trapped BEC system with a finite number of trapped neutral atoms.

In this paper, we treat the infinite-dimensional matrices which appear in the generalized Bogoliubov transformation (GBT). In order to treat the matrices, first, we define the matrices as finite-dimensional ones, e.g., $N \times N$ matrices. Then the limit $N \to \infty$ is taken under an assumption that the limit is well defined, i.e., no divergent term appears in the determinants and products of the matrices. This attitude is justified, because