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General relativistic Boltzmann equation, II: Manifestly covariant treatment

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ABSTRACT

In a preceding article we presented a general relativistic treatment of the derivation of the Boltzmann equation. The four-momenta occurring in this formalism were all on-shell four-momenta, verifying the mass-shell restriction $p^2 = m^2c^2$. Due to this restriction, the resulting Boltzmann equation, although covariant, turned out to be not manifestly covariant. In the present article we switch from mass-shell momenta to off-shell momenta, and thereby arrive at a Boltzmann equation that is manifestly covariant.

1. Introduction

The Boltzmann equation is an equation describing the evolution in time of the one-particle density in position–momentum space. In non-relativistic physics, the one-particle phase–space appears as the direct product of the 3-dimensional physical space and the 3-dimensional momentum space. A point in this phase–space is usually represented by a set of six coordinates $(x^1, x^2, x^3, p^1, p^2, p^3)$.

The general relativistic situation is more complex, since the 3-dimensional physical space has to be replaced by the 4-dimensional space–time M with metric $g_{\mu\nu}(x)$. A first possibility, developed in the preceding article, is to use a 7-dimensional phase–space. A typical point in this space has seven coordinates, namely the four coordinates $x := (x^0 = ct, x^1, x^2, x^3) \in M$ and the three momentum coordinates (p^1, p^2, p^3) on the mass shell $g_{\mu\nu}(x)p^\mu p^\mu = m^2c^2$, $(\mu = 0, 1, 2, 3)$, at $x \in M$.

Notwithstanding this complication, one may extend the usual concept of a one-particle distribution function to the realm of general relativity by defining a one-particle distribution function on a 7-dimensional manifold, and show that this function is a general relativistic scalar. Moreover, we could prove [1] that this general relativistic distribution function obeys a transport equation which may be considered as the general relativistic equivalent of the non-relativistic Boltzmann equation.

Although our approach was covariant, it was not manifestly so, a fact which is related to the use of a momentum space that is 3-dimensional, whereas a truly covariant treatment would require the use of all four components of the four-momentum as a variables; working *on-shell* restricts the 4-dimensional momentum space to a 3-dimensional submanifold.

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Our approach so far used either the three spatial contravariant components (p^1, p^2, p^3) or the three spatial covariant momentum components (p_1, p_2, p_3) as coordinates on the mass shell. An equation satisfied by a function which, apart from the four space–time coordinates, depends on only three momentum components – and not on four – obviously cannot be *manifestly* covariant.

In order to relate our results [2] to earlier ones [3–16] it is necessary to develop a manifestly covariant formalism, and this can only be achieved by using a 4-dimensional momentum space, i.e., by working *off-shell*.

The present article is devoted to this task. Our final results, will turn out to be closely related to, but slightly different from those found in the existing literature. In earlier publications, the Boltzmann equation had been obtained via an educated guess, which resulted in some ambiguities related to the presence or non-presence of the zero component of the momentum in the final equations.

The material of this article is organized as follows.

In Section 2, we switch from the on-shell momentum variables p_μ to off-shell momentum variables p_μ. We introduce an off-shell distribution function $F_*(t, x^i, p_0, p_1, p_2, p_3)$ replacing on-shell distribution $f_*(t, x^i, p_1, p_2, p_3)$ of the preceding article in case covariant momentum variables are used. Furthermore, we derive the manifestly covariant Boltzmann equation (53) for $F_*(t, x^i, p_0, p_1, p_2, p_3)$.

In Section 3 we derive the results for the contravariant case *from* those for the covariant case. An off-shell distribution function $F(t, x^i, p^0, p^1, p^2, p^3)$ takes over the role of $f(t, x^i, p^1, p^2, p^3)$. The resulting equation is the manifestly covariant Boltzmann equation (76) for $F(t, x^i, p^0, p^1, p^2, p^3)$.

In Section 4 the Boltzmann equations (53) and (76) for the off-shell functions F_* and F are rewritten as Eqs. (91) and (110) for the on-shell functions f_* and f.

In the existing literature, one encounters Boltzmann equations containing a distribution function called f, which closely resemble one of the four equations (53), (76), (91) or (110) derived in this article. It is not always clear, however, whether the authors used co- or contravariant momentum variables. Moreover, it is not always made explicit whether momentum variables are on-shell or off-shell. In other words, it is not always clear whether the f encountered in the literature equals F, F_*, f or f_*. This makes it difficult – and sometimes even impossible – to rightly interpret these equations or to compare them to our results.

Appendices A–C give details of the calculations of the main text.

2. Covariant momentum

2.1. Off-shell distribution function F_*

Let us denote a covariant momentum four-vector, like p_μ, which has lower indices, by p_*, and a contravariant vector p^μ by p. In general, a lower asterisk is used to indicate that lower indices are used. Let us now work in an 8-dimensional phase–space, and let us choose the eight coordinates (x, p_*) as independent coordinates in this phase–space. The choice of (x, p) as independent coordinates in an 8-dimensional phase–space will be envisaged later.

If one uses the covariant momentum components p_μ, the contravariant momentum components p^μ are to be considered as functions of (x, p_*). One has

$$p^\mu(x, p_*) = g^{\mu\nu}(x)p_\nu. \tag{1}$$

Inversely, one has

$$p_\mu(x, p) = g_{\mu\nu}(x)p^\nu. \tag{2}$$

We will often suppress the explicit (t, x^i)-dependence. In the preceding article [2] we used these relations to express the zeroth components p^0 and p_0, given in terms of the space–time position $x = (t, x^i)$ and the three-momenta p^i and p_i by

$$p^0(t, x^i, p^i) = \frac{1}{g_{00}(x)}\left\{-g_{0i}(x)p^i + \sqrt{(g_{0i}(x)p^i)^2 - g_{00}(x)(g_{ij}(x)p^ip^j - m^2c^2)}\right\} \tag{3}$$

$$p_0(t, x^i, p_i) = \frac{1}{g^{00}(x)}\left\{-g^{0i}(x)p_i + \sqrt{(g^{0i}(x)p_i)^2 - g^{00}(x)(g^{ij}(x)p_ip_j - m^2c^2)}\right\} \tag{4}$$

[see (I.18) and (I.8)][1] in terms of the components of the three-momentum p_i and p^i. Using (1) and (2) we obtain

$$p^0(t, x^i, p_i) = \sqrt{(g^{0i}(x)p_i)^2 - g^{00}(x)(g^{ij}(x)p_ip_j - m^2c^2)}, \tag{5}$$

$$p_0(t, x^i, p^i) = \sqrt{(g_{0i}(x)p^i)^2 - g_{00}(x)(g_{ij}(x)p^ip^j - m^2c^2)} \tag{6}$$

[see (I.9) and (I.19)].

[1] References to formulae of the preceding article [2] are preceded by the Roman numeral I.