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Thermostatistical properties of a two-parameter generalised quantum group fermion gas

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Abstract

A two-parameter generalized quantum group fermion gas with $SU_{p/q}(2)$-symmetry is discussed, and for low temperatures, its thermostatistical properties obtained by means of a $SU_{p/q}(2)$-covariant fermionic Hamiltonian are investigated in terms of the real independent deformation parameters p and q. The free fermion gas results with two different kinds of undeformed fermionic particle families can be obtained by applying the limit $p = q = 1$.

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1. Introduction

Quantum groups and their associated algebras are deformed versions of the usual Lie groups and Lie algebras, to which they reduce when the deformation parameters (real or complex) are set equal to unity [1]. They have most popularly found applications in several areas of research of theoretical physics such as formulations of noncommutative geometry [2], exactly solvable statistical models [3], and hadronic physics [4]. Furthermore, many studies have been conducted to obtain a physical interpretation of the deformation parameter q by considering a canonical ensemble of q-oscillators [5], which are q-deformations of the bosonic harmonic oscillator algebra. Besides, some possible connections between quantum groups and generalized statistical mechanics have recently been studied [6]. In addition, the statistical and thermodynamical properties of a gas of q-deformed bosons as well as fermions have been extensively investigated in the literature [7].

On the other hand, some studies have been particularly carried out to obtain some possible consequences of introducing quantum group invariance to the systems of deformed bosonic and fermionic particle gases. In these studies, $SU_q(2)$-covariant bosonic and fermionic oscillator algebras play a fundamental role. For instance, the high and low temperature thermodynamics of the one-parameter deformed fermion gas having the symmetry of the quantum

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group $SU_q(2)$ have recently been studied by Ubriaco [8–10]. Among these studies, the high temperature behaviour of the $SU_q(2)$-covariant fermionic gas has provided a new insight into studies of fractional statistics by showing an anyon-like behaviour in three spatial dimensions [8,9]. Recently, the results of Ubriaco's works have been generalised to the two-parameter deformed quantum group boson and fermion gas models whose two-parameter generalised particle algebras are invariant under the quantum group $SU_r(d)$ with $r = q_1/q_2$, $(q_1, q_2) \in R$ [11–16]. However, a different bosonic generalisation has been carried out by considering the two-parameter quantum group $GL_{p,q}(2)$-invariant boson gas defined under the condition $p = q^*$, $(p, q) \in C$ by Jellal [17].

Furthermore, in our recent study [16] we showed that a bosonic version of the present two-parameter $SU_{p/q}(2)$-fermion model exhibits Bose–Einstein condensation for low temperatures in a specific interval of the model deformation parameters q and p. The high-temperature behaviour of the $SU_{p/q}(2)$-fermion model does also have crucial importance [12]. It is shown that this two-parameter fermion model in two spatial dimensions exhibits a remarkably anyonic type of behaviour at some critical values of the deformation parameters q and p. However, it is impossible to obtain a similar behaviour either in the one-parameter deformed $SU_q(2)$-fermion gas model [8] or in the free fermion gas. Such an interesting result has thus motivated us to study the low-temperature thermodynamical and statistical properties of the two-parameter $SU_{p/q}(2)$-covariant fermion model. Some further motivation comes from the recent studies on integrable multiparametric quantum spin chains [18]. For instance, it was shown in Ref. [19] that the quantum algebraic structure of the spin-1/2 XXZ chain with twisted periodic boundary conditions is a two-parameter deformed algebra $SU_{q,t}(2)$.

The main objective of this paper is to study the thermostatistical properties of a fermionic gas having the symmetry of the quantum group $SU_{p/q}(2)$. The results obtained in this way will serve as a two-parameter generalisation related to the low-temperature thermodynamics of earlier quantum group fermion gas studies [8–10]. We start with a fermionic Hamiltonian covariant under the quantum group $SU_{p/q}(2)$. We construct this Hamiltonian from the deformed operators generating a two-parameter deformed $SU_{p/q}(2)$-covariant fermion algebra. Obviously, the undeformed fermion algebra is obtained when both deformation parameters are set to unity. We then investigate the low-temperature (high density) behaviour of such a two-parameter deformed fermion model with $SU_{p/q}(2)$-symmetry. Specifically, by considering the role of the deformation parameters q and p, we discuss the effects of deformation on some thermodynamical characteristics of the present two-parameter $SU_{p/q}(2)$-fermion gas such as internal energy, entropy, and specific heat of the system.

We organise the paper as follows: In Section 2, we give the quantum group covariant two-parameter deformed fermion algebra and discuss its structural and representative properties. In Section 3, we introduce our model described by a $SU_{p/q}(2)$-covariant fermionic Hamiltonian. This leads to the discussion of the thermostatistics of the model obtained via the grand partition function given in Section 4. In particular, in the low-temperature limit, we calculate the average number of particles $\langle N\rangle$, the internal energy U and the entropy S of the $SU_{p/q}(2)$-fermion gas. In the low and high temperature limits, we also find the specific heat of the system in terms of some functions of the deformation parameters p and q. Finally, we discuss possible statistical mechanical implications by invoking a two-parameter quantum group symmetry on a free fermionic gas, and give our conclusions.

2. Quantum group $SU_{p/q}(2)$-fermions

In this section, we briefly review the basic properties of the $SU_q(N)$-fermions and present their two-parameter generalisation. The usual fermionic oscillators satisfy the following anti-commutation relations:

$$\psi_i \psi_j^* + \psi_j^* \psi_i = \delta_{ij}, \quad \psi_i \psi_j + \psi_j \psi_i = 0, \quad \psi_i^* \psi_i = N_i, \quad i, j = 1, 2, \ldots, N, \tag{1}$$

where ψ_i and ψ_i^* are the fermionic annihilation and creation operators, respectively, and N_i is the fermion number operator. These oscillators are covariant under the action of the undeformed group $SU(N)$. The quantum group analogues of these relations are defined as follows [20]:

$$\Psi_j \Psi_i^* = \delta_{ij} - q^{-1} R_{kijl} \Psi_l^* \Psi_k, \tag{2}$$

$$\Psi_l \Psi_k = -q R_{jikl} \Psi_j \Psi_i, \quad i, j = 1, 2, \ldots, N, \tag{3}$$

where the $N^2 \times N^2$ matrix R_{jikl} [2] is

$$R_{jikl} = \delta_{jk}\delta_{il}(1 + (q-1)\delta_{ij}) + (q - q^{-1})\delta_{ik}\delta_{jl}\theta(j - i), \tag{4}$$